Hardy-Hodge decomposition of vector fields on compact Lipschitz hypersurfaces

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Abstract

For $\mathcal{M}$ a compact Lipschitz Riemannian manifold of dimension at least 2, we prove a Helmholtz-Hodge decomposition of tangent L^p vector fields as a sum of a gradient and a divergence free fields; the result holds for restricted range of p around 2, and for all $p \in (1, \infty)$ when $\mathcal{M}$ is VMO-smooth. If, moreover, $\mathcal{M}$ is a compact and connected hypersurface having the local Lipschitz graph property, embedded in $\mathbb{R}^{n+1}$ with the natural metric, we also establish a Hardy-Hodge decomposition of a $\mathbb{R}^{n+1}$-valued vector field of L^p class on $\mathcal{M}$ as the sum of a tangent divergence free field and of two (traces of) harmonic gradients of Hardy class with exponent p, one from inside and one from outside $\mathcal{M}$. The latter holds for restricted range of p, and for all $p \in (1, \infty)$ when $\mathcal{M}$ is C^1-smooth.

Keywords: harmonic gradients, boundary value problems, Helmholtz-Hodge decomposition, Clifford analysis.

1. Notation and preliminaries

sec:DF

Let $\mathbb{R}^k$ denote the Euclidean space of dimension k. We write $x = (x_1, \cdots, x_k)^t$ to indicate the coordinates of $x \in \mathbb{R}^k$, with superscript "t" to mean "transpose". We put $\langle x, y \rangle = \sum_j x_j y_j$, for the scalar product of $x, y \in \mathbb{R}^k$, and $|x| = \langle x, x \rangle^{1/2}$ for the Euclidean norm of x. Hereafter, $B(x, r)$ stands for

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the open ball centered at x of radius r and $S(x,r)$ for the boundary sphere, while $r(B)$ indicates the radius of the ball B. When B is a ball of radius r and λ a strictly positive real number, we put λB for the concentric ball with radius λr. A similar convention holds for cubes. We set $\mathrm{d}(E,F)$ to mean the Euclidean distance between sets $E, F \subset \mathbb{R}^k$, and $\mathrm{diam}E$ to designate the diameter of E. The notation is independent of k, but this will cause no confusion. We put χ_E for the characteristic function of a set E, and use the same letter at different places (*e.g.* "C") to designate various constants.

1.1. Lebesgue and Sobolev spaces

We let m_k denote Lebesgue measure on $\mathbb{R}^k$. If $E \subset \mathbb{R}^k$ is measurable and $p \in [1,\infty]$, we write $L^p(E,\mathbb{R}^m)$ for the familiar Lebesgue space of (equivalence classes of m_k-a.e. coinciding) $\mathbb{R}^m$-valued measurable functions on E whose p-th power is summable, with norm $\|g\|_{L^p(E,\mathbb{R}^m)} = (\int_E |g|^p dm_k)^{1/p}$ (ess. $\sup_E |g|$ if $p=\infty$). If E is open, we set $L^p_{loc}(E,\mathbb{R}^m)$ to consist of functions f whose restriction $f_{|K}$ to K lies in $L^p(K,\mathbb{R}^m)$, for every compact $K \subset E$. When $m=1$, we simply write $L^p(E)$ instead of $L^p(E,\mathbb{R})$.

For $1 \le p < \infty$, we denote the conjugate exponent with a prime superscript: $1/p + 1/p' = 1$. The dual of $L^p(E,\mathbb{R}^m)$ is $L^{p'}(E,\mathbb{R}^m)$ under the pairing

$$(X,Y)_E = \int_E \langle X, Y\rangle dm_k \qquad X \in L^p(E,\mathbb{R}^m),\ Y \in L^{p'}(E,\mathbb{R}^m). \tag{1}$$

pairingOpq

If $\Omega \subset \mathbb{R}^k$ is open, we let $W^{1,p}(\Omega)$ be the Sobolev space of functions lying in $L^p(\Omega)$ together with their first distributional derivatives. It is a Banach space with norm

$$\|g\|_{W^{1,p}(\Omega)} = \left(\|g\|^p_{L^p(\Omega)} + \|\nabla g\|^p_{L^p(\Omega,\mathbb{R}^k)}\right)^{1/p},$$

where $\nabla g = (\frac{\partial g}{\partial x_1}, \cdots, \frac{\partial g}{\partial x_k})^t$ indicates the gradient of g. The space $W^{1,p}_{loc}(\Omega)$ is comprised of functions lying in $L^p_{loc}(\Omega)$ together with their first order derivatives. We let $W^{1,p}_0(\Omega)$ stand for the closure in $W^{1,p}(\Omega)$ of $C^\infty_0(\Omega)$, the space of infinitely differentiable functions with compact support in Ω. We denote by $\dot{W}^{1,p}(\mathbb{R}^k)$ the homogeneous Sobolev space of functions in $L^1_{loc}(\mathbb{R}^k)$ whose distributional derivatives lie in $L^p(\mathbb{R}^k)$. These lie in $W^{1,p}_{loc}(\mathbb{R}^k)$ [15, Theorem 6.74], and if we identify functions which differ by an additive constant, then $\dot{W}^{1,p}(\mathbb{R}^k)$ is a Banach space with norm $\|g\|_{\dot{W}^{1,p}(\mathbb{R}^k)} = \|\nabla g\|_{L^p(\mathbb{R}^k,\mathbb{R}^k)}$. When $1 < p < \infty$, it is in fact the closure with respect to $\|.\|_{\dot{W}^{1,p}(\mathbb{R}^k)}$ of smooth functions with compact support; this follows, *e.g.* from [36, Thm. 1]. We write

$W^{1,p}(\Omega,\mathbb{R}^m)$ (resp. $W^{1,p}_{loc}(\Omega,\mathbb{R}^m)$, ...) for corresponding spaces of $\mathbb{R}^m$-valued maps. The space $W^{2,p}(\Omega)$ consists of functions in $L^p(\Omega)$ whose derivatives of the first order lie in $W^{1,p}(\Omega)$. The local version is designated by $W^{2,p}_{loc}(\Omega)$.

1.2. Lipschitz functions

A map $f : E \to \mathbb{R}^m$, $E \subset \mathbb{R}^k$, is Lipschitz if $|f(x) - f(y)| \leq C|x - y|$ for some constant C and all $x, y \in E$. The smallest C for which this holds is the Lipschitz constant of f, denoted by $\mathfrak{c}_f$. It is easily checked that f maps sets of Lebesgue measure zero to sets of Lebesgue measure zero if $m \geq k$. More generally, if $B \subset E$ is Borel, then $\mathcal{H}^s(f(B)) \leq \mathfrak{c}_f^s \mathcal{H}^s(B)$ where $\mathcal{H}^s$ indicates s-dimensional Hausdorff measure, see *e.g.* [21, 56] for a definition of Hausdorff measure and the fact that $\mathcal{H}^k$ essentially coincides with m_k on $\mathbb{R}^k$. If, in addition, f is injective and its inverse $f^{-1} : \mathrm{Im}\, f \to E \subset \mathbb{R}^k$ is also Lipschitz, then we say that f is bi-Lipschitz. In this case, f is a homeomorphism onto its image, and if E is open then necessarily $m \geq k$, by invariance of the domain [42, chap. 10, sect. 62]. If each $x \in E$ has a relative neighborhood O_x in E such that the restriction $f_{|O_x}$ is Lipschitz, we say that f is locally Lipschitz.

When $\Omega \subset \mathbb{R}^k$ is open and $f : \Omega \to \mathbb{R}^m$ is Lipschitz, a result by Rademacher asserts that f is differentiable m_k-a.e on Ω. Clearly, $\|Df(x)\| \leq \mathfrak{c}_f$, where $Df(x)$ indicates the derivative of f at x and $\|.\|$ the operator norm. Moreover, the distributional derivatives of a Lipschitz function agree with the pointwise derivatives m_k-a.e., hence the space of $\mathbb{R}^m$-valued Lipschitz functions on a bounded open set $\Omega \subset \mathbb{R}^k$ coincides with $W^{1,\infty}(\Omega,\mathbb{R}^m)$, see [51, Ch. V, Sec. 6 and Ch. VIII, Thms. 1&3].

Let $\Omega \subset \mathbb{R}^k$ be open and $f : \Omega \to \mathbb{R}^m$ be Lipschitz, with $m \geq k$. At each $x \in \Omega$ where f is differentiable, we let $J_k f(x)$ be the k-dimensional Jacobian, namely the square root of the sum of the squares of the $k \times k$ determinants of $Df(x)$, when the latter is identified with its matrix in the canonical bases of $\mathbb{R}^k$ and $\mathbb{R}^m$ (the familiar Jacobian matrix). If f is also injective, then the area formula [21, Thm. 3.2.3] and the extendability of Lipschitz functions [21, Thm. 2.10.43] together imply that, for every integrable function u on Ω,

$$\int_\Omega u(x) J_k f(x) dm_k(x) = \int_{f(\Omega)} u(f^{-1}(y))\, d\mathcal{H}^k(y). \qquad (2) \quad \boxed{\texttt{area}}$$

Note that when $m = k$, the identity (2) reduces to the standard change of variable formula for Lipschitz reparametrizations.

Let $\Omega \subset \mathbb{R}^k$ be open and $f : \Omega \to \mathbb{R}^m$ be bi-Lipschitz. For $x \in \Omega$, $w \in \mathbb{R}^k \backslash \{0\}$ and $t > 0$ small enough, we obviously have that

$$|w| = \frac{|f^{-1}(f(x+tw)) - f^{-1}(f(x))|}{|f(x+tw) - f(x)|} \, \frac{|f(x+tw) - f(x)|}{t},$$

therefore we get on letting $t \to 0$ that if f is differentiable at x, then

$$\frac{|w|}{\mathfrak{c}_{f^{-1}}} \leq |Df(x)(w)| \leq \mathfrak{c}_f |w|. \qquad w \in \mathbb{R}^k, \tag{3}$$ diffpi

It follows in particular from (3) that Df is injective. If $f : \Omega \to \Omega'$ is a bi-Lipschitz homeomorphism between open sets in $\mathbb{R}^k$ and $g : \Omega' \to \mathbb{R}^m$ is Lipschitz, then the chain rule holds: $D(g \circ f)(x) = Dg(f(x))Df(x)$ for a.e. $x \in \Omega$; it is so because the set of $x \in \Omega$ such that g is not differentiable at $f(x)$ has Lebesgue measure zero.

BMOgen *1.3. BMO and VMO functions*

For $\Omega \subset \mathbb{R}^k$ an open set, the space $BMO(\Omega)$ of functions with bounded mean oscillation on Ω consists of those $h \in L^1_{loc}(\Omega)$ such that

$$\sup_{\overline{Q} \subset \Omega} \inf_{b \in \mathbb{R}} \frac{1}{m_k(Q)} \int_Q |h(x) - b| \, dm_k(x) < \infty, \tag{4}$$ defBMOTc

where the supremum is over all open cubes Q with closure $\overline{Q} \subset \Omega$. If we let

$$h_Q := \frac{1}{m_k(Q)} \int_Q h(x) \, dm_k(x)$$

be the mean of h on Q, we get that $\int_Q |h - h_Q| \, dm_k \leq 2 \int_Q |h - b| \, dm_k$ for every $b \in \mathbb{R}$, therefore (4)is equivalent to

$$\|h\|_{BMO(\Omega)} := \sup_{\overline{Q} \subset \Omega} \frac{1}{m_k(Q)} \int_Q |h(x) - h_Q| \, dm_k(x) < \infty. \tag{5}$$ defBMOT

One may restrict to cubes with sides parallel to the axes, or replace cubes by balls, as this will define equivalent quantities (*i.e.* whose ratio with (5) is bounded above and below by strictly positive constants). This follows from the work in [47, 37, 50]. Note that $\| \cdot \|_{BMO(\Omega)}$ is a norm modulo additive constants only.

The space $VMO(\Omega)$ of functions with vanishing mean oscillation is the closed subspace of $BMO(\Omega)$ comprised of those h for which

$$\lim_{\varepsilon\to 0}\sup_{\substack{\overline{Q}\subset\Omega\\ m_k(Q)<\varepsilon}}\frac{1}{m_k(Q)}\int_Q |h(x)-h_Q|\,dm_k(x)=0. \tag{6}$$

defVMO

The space $VMO_{loc}(\Omega)$ consists of functions $h:\Omega\to\mathbb{R}$ whose restriction to any relatively compact open subset Ω' lies in $VMO(\Omega')$. It is equivalent to require that each $x\in\Omega$ has an open neighborhood O_x with $h_{|O_x}\in VMO(O_x)$. We denote by $VMO^{1,\infty}(\Omega)$ the subspace of $W^{1,\infty}(\Omega)$ comprised of functions whose first order derivatives lie in $VMO(\Omega)$. The space $VMO^{1,\infty}_{loc}(\Omega)$ consists of functions in $W^{1,\infty}_{loc}(\Omega)$ whose derivatives are in $VMO_{loc}(\Omega)$. It is elementary that $L^\infty(\Omega)\cap VMO(\Omega)$ is an algebra. So, by Leibnitz's rule, $VMO^{1,\infty}(\Omega)$ and $VMO^{1,\infty}_{loc}(\Omega)$ are also algebras.
We write $BMO(\Omega,\mathbb{R}^k)$, $VMO(\Omega,\mathbb{R}^k)$ and so on for $\mathbb{R}^k$-valued functions whose components lie in the space indicated.

defgen

1.4. Compact Lipschitz Riemannian manifolds

A Lipschitz manifold of dimension n is a second countable Hausdorff topological space $\mathcal{M}$ equipped with an open cover $\{V_l\}$, each element of which is homeomorphic to an open subset of $\mathbb{R}^n$ under a map $\psi_l:V_l\to\mathbb{R}^n$, in such a way that changes of charts $\psi_{l_1}\circ\psi_{l_2}^{-1}:\psi_{l_2}(V_{l_1}\cap V_{l_2})\to\psi_{l_1}(V_{l_1})$ are locally Lipschitz. We call $\cup_l(V_l,\psi_l)$ an atlas for $\mathcal{M}$, and two atlases are equivalent if their union is again an atlas. The collection of all atlases equivalent to the initial one defines the corresponding Lipschitz structure on $\mathcal{M}$. It is no loss of generality to assume that $\mathcal{M}$ is embedded into $\mathbb{R}^m$ for some $m\geq n$, moreover we may pick the atlas so that the charts $\psi_l:V_l\to\mathbb{R}^n$, as well as the parametrizations $\psi_l^{-1}:\psi_l(V_l)\to V_l\subset\mathbb{R}^m$, are locally Lipschitz; in fact $m=n(n+2)$ will do, see [43, Thm.4.9]. In particular, $\mathcal{M}$ is a metric space with metric induced by Euclidean distance on $\mathbb{R}^m$, which is compatible with the local metrics induced by the Euclidean sets $\psi_l(V_l)$, see [43].
We assume throughout that $n\geq 2$ and restrict ourselves to compact $\mathcal{M}$, hence we may refine a given atlas (V_l) (*i.e.* replace it by another atlas (U_j,ϕ_j) such that each U_j is contained in some V_l with $\phi_j=\psi_{l|U_j}$) to get an equivalent atlas (U_j,ϕ_j), where j ranges over a finite set $\{1,\cdots,N\}$ and $\phi_j:U_j\to B_j$ is bi-Lipschitz onto a bounded open set $B_j\subset\mathbb{R}^n$ for each j. We call such an atlas a B-atlas, for short. Clearly, we may assume in addition that B_j is

smooth and connected (*e.g.* an open ball). Note that a B-atlas is Lipschitz, meaning that changes of charts are Lipschitz, not just locally Lipchitz.
A point $x \in \mathcal{M}$ is said to be singular if there is $j \in \{1, \cdots, N\}$ such that $x \in U_j$ and $\phi_j^{-1} : B_j \to \mathbb{R}^m$ is not differentiable at $\phi_j(x)$. A point which is not singular is called regular. The set of regular points depends on the atlas, not just on the Lipschitz structure. Below, we ignore this dependence, for our results will not involve a particular atlas. Accordingly, we (improperly) denote the set of regular points by $\operatorname{Reg} \mathcal{M}$ and put $\operatorname{Reg} B_j = \phi_j(\operatorname{Reg} \mathcal{M} \cap U_j)$. The set of singular points is small in that its image in any chart has Lebesgue measure zero, by Rademacher's theorem. From (3), we get that $D\phi_j^{-1}(y)$ is injective for $y \in \operatorname{Reg} B_j$.
By definition, the tangent space $T_x\mathcal{M} \subset \mathbb{R}^m$ at $x \in \mathcal{M}$ is $\{0\}$ if x is singular, and $T_x\mathcal{M} = \operatorname{Im} D\phi_j^{-1}(\phi_j(x))$ if $x \in U_j \cap \operatorname{Reg} \mathcal{M}$. In the latter case, each $X \in T_x\mathcal{M}$ has a local representative in the chart (U_j, ϕ_j), which is the unique $v \in \mathbb{R}^n$ such that $X = D\phi_j^{-1}(\phi_j(x))v$. A map $g : \mathcal{M} \to \mathbb{R}^k$ is said to be differentiable at $x \in \operatorname{Reg} \mathcal{M}$ if $g \circ \phi_j^{-1}$ is differentiable at $\phi_j(x)$, and the derivative $Dg(x) : T_x\mathcal{M} \to \mathbb{R}^k$ is defined by $Dg(x)(X) = D(g \circ \phi_j^{-1})(\phi_j(x))v$, with v the local representative of X. By the chain rule, the definitions do not depend on j such that $x \in U_j$, and if we use another B-atlas they will agree with the present ones at any point which is regular for both atlases. Note that $\dim T_x\mathcal{M} = n$ at regular points. Lipschitz-smooth partitions of unity subordinated to an open cover of $\mathcal{M}$ can be constructed as in the smooth case.
A Riemannian metric Γ assigns to each $x \in \mathcal{M}$ a positive definite scalar product $\Gamma_x : T_x\mathcal{M} \times T_x\mathcal{M} \to \mathbb{R}$ such that, for each $j \in \{1, \cdots, N\}$ and $1 \leq i, k \leq n$, the local metric tensor $g_{i,k}^{(j)}$ defined a.e. on B_j by

$$g_{i,k}^{(j)}(y) := \Gamma_x \left(\frac{\partial \phi_j^{-1}}{\partial y_i}(y) \, , \, \frac{\partial \phi_j^{-1}}{\partial y_k}(y) \right), \qquad y = (y_1, \cdots, y_n)^t \in \operatorname{Reg} B_j, \quad (7)$$

defMTg

is Lebesgue measurable and, for some constants $c_1, c_2 > 0$ independent of x, we have the inequality:

$$c_1|X|^2 \leq \Gamma_x(X, X) \leq c_2|X|^2, \qquad X \in T_x(\mathcal{M}). \quad (8)$$

RgM

At singular points, Γ_x is the trivial bilinear form on the zero vector space. For $X, Y \in T_x\mathcal{M}$ with $x \in \operatorname{Reg} \mathcal{M} \cap U_j$, the scalar product $\Gamma_x(X, Y)$ can be written as a symmetric bilinear form with matrix $(g_{i,k}^{(j)})$ in terms of the local representatives at $y = \phi_j(x)$:

$$\Gamma_x(X,Y) = v^t(g_{i,k}^{(j)}(y))w, \qquad X = D\phi_j^{-1}(y)v, \qquad Y = D\phi_j^{-1}(y)w. \tag{9}$$ locmetg

The determinant of $(g_{i,k}^{(j)})$ will be denoted by $g^{(j)}$. Since ϕ_j^{-1} is bi-Lipschitz, (3) and (8) together imply the existence of constants c, C such that

$$0 < c \le g^{(j)} \le C, \qquad 0 < cI_n \le (g_{i,k}^{(j)}) \le CI_n, \tag{10}$$ detlocg

where I_n is the identity matrix of size $n \times n$ and the second set of inequalities is understood between symmetric matrices. Alternatively, we could postulate (10) and then (8) would hold. Since there are finitely many charts U_j, the constants c, C in (10) can be made independent of j.
Associated with a Riemannian metric is the volume measure σ on $\mathcal{M}$, whose image under ϕ_j is, for each j, absolutely continuous with respect to m_n on B_j with density $\sqrt{g^{(j)}}$. To check that it exists, pick a Lipschitz partition of unity (φ_j) subordinated to $(U_j)_{1\le j\le N}$ and, for a continuous $f : \mathcal{M} \to \mathbb{R}$, let

$$\int_{\mathcal{M}} f d\sigma = \sum_j \int_{B_j} \Big((\varphi_j f) \circ \phi_j^{-1}(y)\Big) \sqrt{g^{(j)}}(y)\ dm_n(y). \tag{11}$$ integralsg

By the chain rule and the change of variable formula, (11) does not depend on the B-atlas nor the partition of unity we choose, and defines a bounded linear form on continuous functions for the sup-norm, thanks to (10). Thus, by the Riesz representation theorem, σ is a Radon measure. We denote by $L^p_\sigma(\mathcal{M})$ or $L^p_\sigma(\mathcal{M}, \mathbb{R}^\ell)$ the corresponding Lebesgue spaces, and we drop the subscript σ if the Riemannian metric is understood. Clearly, $E \subset \mathcal{M}$ is negligible for σ if and only if $\phi_j(E \cap U_j)$ has Lebesgue measure zero for all j. Moreover, $h \in L^1_\sigma(\mathcal{M})$ if and only if $\sqrt{g^{(j)}}(h \circ \phi_j^{-1}) \in L^1(B_j)$ for each j, that is if and only if $h \circ \phi_j^{-1} \in L^1(B_j)$, in view of (10). We often omit the superscript (j) and simply write $g_{i,k}$ or g when a single chart is involved.
For example, when Γ is the Euclidean scalar product in $\mathbb{R}^m$, we get on identifying $D\phi_j^{-1}$ with the Jacobian matrix that $(g_{i,k}) = (D\phi_j^{-1})^t D\phi_j^{-1}$, hence $g = (J_n\phi_j^{-1})^2$ by the Binet-Cauchy formula. In this case, we see from (2) that $\sigma = \mathcal{H}^n\lfloor\mathcal{M}$, the restriction to $\mathcal{M}$ of n-dimensional Hausdorff measure.
The volume measure σ is doubling, meaning that for some constant C_0:

$$\sigma(B(\xi,r)\cap\mathcal{M}) \le C_0\,\sigma(B(\xi,r/2)\cap\mathcal{M}) < +\infty, \qquad \xi \in \mathcal{M}, \quad r > 0, \tag{12}$$ doubling

where $B(\xi, r)$ indicates a Euclidean ball in $\mathbb{R}^m$. Indeed, by the compactness of $\mathcal{M}$, there is $\delta > 0$ such that every relative ball $B(\xi, \delta) \cap \mathcal{M}$, $\xi \in \mathcal{M}$, is included in some U_j. Then $B(\phi_j(\xi), \delta/\mathfrak{c}_{\phi_j^{-1}}) \subset \phi_j(U_j)$, and for $r < \delta/(\mathfrak{c}_{\phi_j}\mathfrak{c}_{\phi_j^{-1}})$:

$$B(\phi_j(\xi), r/\mathfrak{c}_{\phi_j^{-1}}) \subset \phi_j(B(\xi, r) \cap \mathcal{M}) \subset B(\phi_j(\xi), r\mathfrak{c}_{\phi_j}) \subset \phi_j(U_j), \qquad (13)$$ incB

where we used that $\mathfrak{c}_{\phi_j}\mathfrak{c}_{\phi_j^{-1}} \geq 1$. Letting $r_0 := \min_{1\leq j\leq N} \delta/(\mathfrak{c}_{\phi_j}\mathfrak{c}_{\phi_j^{-1}})$, we deduce from (10), (11) and (13) that

$$c' r^n \leq \sigma(B(\xi, r) \cap \mathcal{M}) \leq C' r^n, \qquad \xi \in \mathcal{M}, \qquad r < r_0, \qquad (14)$$ Ahlfors

where c', C' depend on n, δ, $\max_{1\leq j\leq N}\{\mathfrak{c}_{\phi_j}, \mathfrak{c}_{\phi_j^{-1}}\}$ and the constants c, C in (10). Now, (12) follows at once from (14) and the fact that $\sigma(\mathcal{M}) < \infty$. Because σ is regular, Equation (14) itself expresses that the n-dimensional manifold $\mathcal{M}$ is Ahlfors-David regular in $\mathbb{R}^m$, see [31, Sec. 1]. The doubling property makes $\mathcal{M}$ a space of homogeneous type when equipped with the Euclidean distance induced by $\mathbb{R}^m$ and the volume measure σ, see [9].
Differential forms can be defined as in the smooth case: a 0-form is a real-valued function on $\mathcal{M}$ and, for $k \geq 1$, a k-form ω associates to each $x \in \mathcal{M}$ an alternating k-linear map $\omega(x) : (T_x\mathcal{M})^k \to \mathbb{R}$. In this paper, we only deal with vector fields and we could do away with forms entirely. However, vector fields correspond to 1-forms under the identification $T_x\mathcal{M} \sim T_x^*\mathcal{M}$ *via* Γ_x, and in order to connect Sobolev spaces of functions on $\mathcal{M}$, defined in Section 2, with more general spaces defined in terms of forms classically found in the literature[53, 24], we briefly discuss them.
The local representative of a k-form ω in the chart $\phi_j : U_j \to B_j$ is the k-form $(\phi_j^{-1})^*(\omega)$ on B_j which is the pullback of ω under ϕ_j^{-1} at regular points:

$$(\phi_j^{-1})^*(\omega)(y) := \omega(\phi_j^{-1}(y)) \circ (D\phi_j^{-1}(y) \times \cdots \times D\phi_j^{-1}(y)), \quad y \in \operatorname{Reg} B_j, \; (15)$$ replocc

where the product map has k factors. We set $(\phi_j^{-1})^*(\omega) = 0$ if $y \in B_j \setminus \operatorname{Reg} B_j$. Rearranging (15), we get an expression of the form

$$(\phi_j^{-1})^*(\omega)(y) = \sum_{i_1 < i_2, \cdots, < i_k} a_{i_1,\cdots,i_k}^{\{\phi_j\}}(y) \;\, dy_{i_1} \wedge \cdots \wedge dy_{i_k}, \qquad y \in \operatorname{Reg} B_j, \; (16)$$ coefflocf

where the dy_i are dual to the canonical basis of $\mathbb{R}^n$ and the coefficients $a_{i_1,\cdots,i_k}^{\{\phi_j\}}$ are functions on B_j that transform naturally under changes of coordinates.

We identify k-form which coincide σ-a.e. or, equivalently, whose coefficients agree m_n-a.e. on B_j. We say that ω is of L^p-class if the $a^{\{\phi_j\}}_{i_1,\cdots,i_k}$ in (16) lie in $L^p(B_j)$ for each j.

Integrating n-forms goes as in the smooth case. Namely, given a n-form $\mu(y) = a(y)dy_1 \wedge \cdots \wedge dy_n$ on B_j with $a \in L^1(B_j)$, we set $\int_{B_j} \mu = \int_{B_j} a\, dm_n$. Then, if ω is a n-form of L^1-class on $\mathcal{M}$, we pick a Lipschitz partition of unity (φ_j) subordinated to the U_j and we set $\int_{\mathcal{M}} \omega = \sum_{j=1}^N \int_{B_j} (\phi_j^{-1})^*(\varphi_j \omega)$. The definitions are independent from the B-atlas and the partition of unity we use, thanks to the change of variable formula.

For $0 \leq k < n$, we say that a k-form ω of L^p-class on $\mathcal{M}$ is p-flat if there is a $k+1$-form $d\omega$ of L^p-class (the distributional exterior derivative of ω) such that, for all $j \in \{1, \cdots, N\}$,

$$\int_{B_j} (\phi_j^{-1})^*(\omega) \wedge d\mu_j = (-1)^{k+1} \int_{B_j} (\phi_j^{-1})^*(d\omega) \wedge \mu_j \qquad (17) \quad \boxed{\texttt{extderSobf}}$$

whenever μ_j is a smooth $(n-k-1)$-form compactly supported on B_j. Here, the exterior derivative $d\mu_j$ is the usual one and (17) defines $(\phi_j^{-1})^*(d\omega)$ as a $(n-k-1)$-current on B_j; the assumption that ω is p-flat means that this current is the pullback under ϕ_j^{-1} of a $(k+1)$-form of L^p-class on $\mathcal{M}$, which is called $d\omega$. This definition is consistent, for on $\phi_{j_2}(U_{j_1} \cap U_{j_2})$ it holds that

$$d\Big((\phi_{j_1} \circ \phi_{j_2}^{-1})^*\Big((\phi_{j_1}^{-1})^*(\omega)\Big)\Big) = (\phi_{j_1} \circ \phi_{j_2}^{-1})^*\Big((\phi_{j_1}^{-1})^*(d\omega)\Big), \qquad (18) \quad \boxed{\texttt{consflat}}$$

see [55, Thm. 9C] when $p = \infty$ and [24, Thm. 2.2] for $1 \leq p \leq \infty$, compare also [53, Prop.1.2]. Thus, $d\omega$ does not depend on the B-atlas we choose. By convention, a n-form of L^p-class is p-flat with zero exterior derivative. With the present definition, the ∞-flat forms are the usual flat forms on $\mathcal{M}$, see [55, 24, 53]. Now, if $f : \mathcal{M} \to \mathbb{R}$ is a function, it is easy to see that it is p-flat (as a zero form) if and only if $f \circ \phi_j^{-1} \in W^{1,p}(B_j)$ for all j, and that $(\phi_j^{-1})^*(df) = \sum_i (\partial(f \circ \phi_j^{-1})/\partial y_i) dy_i$.

$\boxed{\texttt{VMOSM}}$ 1.5. *VMO-smooth manifolds*

We say a manifold $\mathcal{M}$ of dimension n is VMO-smooth if there is an atlas $\cup_l (V_l, \psi_l)$ such that changes of charts $\psi_{l_1} \circ \psi_{l_2}^{-1} : \psi_{l_2}(V_{l_1} \cap V_{l_2}) \to \psi_{l_1}(V_{l_1})$ lie in $VMO^{1,\infty}_{loc}(\psi_{l_2}(V_{l_1} \cap V_{l_2}), \mathbb{R}^n)$ for all l_1, l_2. We call $\cup_l (V_l, \psi_l)$ a VMO-smooth atlas, and two such atlases are equivalent if their union is again a VMO-smooth atlas. The collection of all VMO-smooth atlases equivalent to the initial one defines the corresponding VMO-smooth structure on $\mathcal{M}$.

VMO-smooth manifolds are smoother than Lipschitz manifolds, but a little rougher than C^1-manifolds. Being a particular case of Lipschitz manifold, a VMO-smooth manifold has a Lipschitz embeding into $\mathbb{R}^m$ for some $m \leq n(m+2)$, but increasing m as much as $(n+1)^2$ if necessary, we can ensure that $\mathcal{M}$ embeds in $\mathbb{R}^m$ with a VMO-smooth atlas $\cup_l(V_l, \psi_l)$ such that the ψ_l are locally bi-Lipschitz and the parametrizations ψ_l^{-1} lie in $VMO^{1,\infty}_{loc}(\psi_l(V_l), \mathbb{R}^m)$, see Lemma 7.4. When $\mathcal{M}$ is compact, any such atlas can be refined into a B-atlas $\cup_j(U_j, \phi_j)_{1\leq j\leq N}$ such that $\phi_{j_1} \circ \phi_{j_2}^{-1} : \phi_{j_2}(U_{j_1} \cap U_{j_2}) \to \phi_{j_1}(U_{j_1})$ lies in $VMO^{1,\infty}(\phi_{j_2}(U_{j_1} \cap U_{j_2}), \mathbb{R}^n)$ (not just $VMO^{1\infty}_{loc}(\phi_{j_2}(U_{j_1} \cap U_{j_2}), \mathbb{R}^n)$) and ϕ_j^{-1} in $VMO^{1,\infty}(\phi_j(U_j), \mathbb{R}^m)$. We call it a VMO-smooth B-atlas.
If $\mathcal{M}$ is VMO-smooth with Riemannian metric Γ, and for some VMO-smooth atlas $\cup_l(V_l, \psi_l)$ the local metric tensor lies in $VMO_{loc}(\psi_l(V_l), \mathbb{R}^{n\times n})$ for each l, then the same is true of any VMO-smooth atlas, see Lemma 7.5. Thus, it makes sense to call such a Γ a VMO metric on $\mathcal{M}$. If, in addition, $\mathcal{M}$ is compact, then every VMO-smooth B-atlas can be refined into another one, say $\cup_j(U_j, \phi_j)_{1\leq j\leq N}$, such that $(g^{(j)}_{i,k}) \in VMO(\phi_j(U_j), \mathbb{R}^{n\times n})$ for each j.

LHGP *1.6. Lipschitz hypersurfaces with the local graph property*

In Sections 4, 5 and 6, we restrict to the case of a compact and connected hypersurface $\mathcal{M} \subset \mathbb{R}^{n+1}$ having the *local Lipschitz graph property*, meaning that it is locally the graph of a Lipschitz function. This is a particular type of Lipschitz manifold for which a B-atlas (U_j, ϕ_j) may be chosen so that $\phi_j : U_j \to B_j$ is of the form $(P_n \circ L_j)_{|U_j}$, where L_j is a linear isometry of $\mathbb{R}^{n+1}$ and $P_n : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection onto the first n-components, while $\phi_j^{-1} = L_j^{-1} \circ (I_n, \Psi_j)^t$ for some Lipschitz map $\Psi_j : B_j \to \mathbb{R}$. We call such an atlas a *G-atlas*, for short. Note that ϕ_j extends naturally to a globally defined linear map $\mathbb{R}^{n+1} \to \mathbb{R}^n$. We may assume in addition that B_j is smooth and connected (for instance a ball in $\mathbb{R}^n$), but such extra assumptions will be made explicit.
Being a closed hypersurface, $\mathcal{M}$ is orientable [29, Cor. 3.46]. If in addition it is connected, then its complement has two components: the interior denoted by Ω^+, which is bounded, and the exterior denoted by Ω^-, which is unbounded [29, Cor. 3.45]. For $\cup_j(U_j, \phi_j)$ a G-atlas on $\mathcal{M}$, we make the convention that for some numbers $a_j < b_j$ and $\varepsilon_j > 0$ such that $a_j < \inf_{y\in B_j} \Psi_j(y) - \varepsilon_j$ and $b_j > \sup_{y\in B_j} \Psi_j(y) + \varepsilon_j$, the image by L_j^{-1} of the hypograph $\{(x,t) : x \in B_j, a_j \leq t < \Psi_j(x)\}$ (resp. epigraph $\{(x,t) : x \in B_j, \Psi_j(x) < t \leq b_j\}$) is included in Ω^+ (resp. Ω^-) for every j.

This can always be achieved by composing L_j with a rotation and Ψ_j with an affine transformation.
The open set $\mathcal{C}_j := L_j^{-1}(B_j \times (a_j, b_j))$, whose intersection with $\mathcal{M}$ is U_j, will be called a *coordinate cylinder* (over U_j). We orient its axis in the direction of the unit vector $v_j^- := L_j^{-1}(0, \cdots, 0, 1)^t$ pointing towards Ω^-, and we set $v_j^+ = -v_j^-$ for the opposite unit vector pointing towards Ω^+. We shall assume that the bases $B_j \times \{a_j\}$ and $B_j \times \{b_j\}$ of the coordinate cylinders have closure contained in Ω^+ and Ω^-, respectively. This can be ensured by truncating the cylinders to slightly shorter length and adding a small constant to Ψ_j.
The local Lipschitz graph property of $\mathcal{M}$ is equivalent to the fact that $\Omega^\pm$ has the so-called uniform cone property; *i.e.*, each boundary point (that is: each point of $\mathcal{M}$) has a neighborhood V such that the translate of a fixed positive truncated circular cone at any point of $V \cap \Omega^\pm$ is contained in $\Omega^\pm$, see [27, Thm. 1.2.2.2]. In this connection, the following construction is useful. For $y, z \in \mathbb{R}^{n+1}$ with $|z| = 1$, and $\theta \in (0, \pi/2)$, we put $C_{\theta,z}(y)$ for the open right circular positive cone with vertex y, axis directed along z and aperture angle 2θ, cut off to some suitable length. For simplicity, we do not make the length explicit in the notation, but we shall indicate how to choose it when needed.
For (U_j, ϕ_j) a G-atlas and V_j an open cover of $\mathcal{M}$ such that $\overline{V}_j \subset U_j$ (such a cover can always be found), then

$$\overline{C_{\theta,v_j^\pm}(\xi)} \setminus \{\xi\} \subset C_{\theta+\epsilon,v_j^\pm}(\xi) \subset \mathcal{C}_j \cap \Omega^\pm, \qquad \xi \in V_j, \tag{19}$$

inclucone

provided that $\theta \in (0, \pi/2)$ and $\epsilon > 0$ satisfy $\tan\theta < \tan(\theta + \epsilon) < 1/M$ with $M := \max_j \mathfrak{c}_{\Psi_j}$, while $C_{\theta,v_j^\pm}(\xi)$ is cut off to length strictly less than $\mathrm{d}(\phi_j(\overline{V}_j), \mathbb{R}^n \setminus B_j)/\tan(\theta + \epsilon)$. We call $C_{\theta,v_j^\pm}$ a *natural cone* for $\Omega^\pm$ (relative to U_j and V_j), meaning that its axis is $v_j^\pm$ and its aperture is strictly less than $2\tan^{-1}(1/M)$, while it is truncated to sufficiently small length that (19) holds for some $\epsilon > 0$.
Note that when $\xi \in V_j$ and $r > 0$ is small enough, the cone $C_{\theta,v_j^\pm}(x)$ is included in $\Omega^\pm$ for each $x \in B(\xi, r) \cap \Omega^\pm$, and since the V_j cover $\mathcal{M}$ this evidences the uniform cone property.
For (U_j, ϕ_j) a G-atlas, identifying $D\phi_j^{-1}(x)$ with its $(n+1) \times n$ matrix in the canonical bases of $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$, we get if ψ_j is differentiable at $y \in B_j$ that

$$D\phi_j^{-1}(y) = L_j^{-1}\Big(I_n\, ,\, \nabla\Psi_j(y)\Big)^t, \tag{20}$$

laformphi

hence $x \in \mathrm{Reg}\mathcal{M}$ if and only if ψ_j is differentiable at $\phi_j(x)$ whenever $x \in U_j$.

By restriction to $T_x\mathcal{M}$, the Euclidean scalar product in $\mathbb{R}^{n+1}$ induces a Riemannian metric on $\mathcal{M}$ whose local metric tensor in a chart is a rank-1 perturbation of the identity:

$$\begin{aligned}(g_{i,k})(y) &= \big(D\phi_j^{-1}(y)\big)^t\, D\phi_j^{-1}(y)\\ &= I_n + \nabla\Psi_j(y)\,\big(\nabla\Psi_j(y)\big)^t, \quad y\in\operatorname{Reg} B_j.\end{aligned} \tag{21}$$ format

Using (21), the determinant of $(g_{i,k})$ is easily computed to be

$$g(y) = 1 + |\nabla\Psi_j(y)|^2, \quad y\in\operatorname{Reg} B_j, \tag{22}$$ calculg

so the volume measure $\sigma = \mathcal{H}^n\lfloor\mathcal{M}$ has image $\sqrt{1+|\nabla\Psi_j|^2}\,dm_n$ under ϕ_j. The outward unit normal n_x to $\mathcal{M}$ at x can be defined σ-a.e. in measure-theoretic terms: recall the measure-theoretic normal to a set $E\subset\mathbb{R}^{n+1}$ at x is the only unit vector v (if it exists) such that

$$\lim_{r\to 0} r^{-(n+1)} m_{n+1}\big(B(x,r)\cap\{y\notin E,\ (y-x).v<0\}\big) = 0 \tag{23}$$ mtn1

and

$$\lim_{r\to 0} r^{-(n+1)} m_{n+1}\big(B(x,r)\cap\{y\in E,\ (y-x).v>0\}\big) = 0. \tag{24}$$ mtn2

Now, the local graph property implies that the measure theoretic normal n_x to Ω^+ at $x\in U_j$ exists when Ψ_j is differentiable at $y=\phi_j(x)$, and is equal to $L_j^{-1}(-\nabla\Psi_j(y)^t,1)^t/\sqrt{1+|\nabla\Psi_j(y)|^2}$, see [56, Rem. 5.8.3]. Conversely, if (23), (24) hold with $E=\Omega^+$ at $x\in U_j$, it is not difficult to see that the Lipschitz function Ψ_j must be differentiable at $\phi_j(x)$ and that $\operatorname{Im} D\Psi_j(\phi_j(x))$ is orthogonal to v. This makes for an intrinsic definition of $\operatorname{Reg}\mathcal{M}$ in this case.

By definition (see [56, 5.8.4]), the measure-theoretic boundary of a set $E\subset\mathbb{R}^{n+1}$ consists of those $x\in\mathbb{R}^{n+1}$ such that

$$\limsup_{\rho\to 0}\frac{m_{n+1}(B(x,\rho)\cap E)}{m_{n+1}(B(x,\rho))}>0 \quad\text{and}\quad \limsup_{\rho\to 0}\frac{m_{n+1}(B(x,\rho)\setminus E)}{m_{n+1}(B(x,\rho))}>0. \tag{25}$$ defthb

We record for later use that the measure theoretic boundary of $\Omega^\pm$ is $\mathcal{M}$. This is obvious, for instance from the uniform cone property.

We say that a compact hypersurface $\mathcal{M}\subset\mathbb{R}^{n+1}$ has the local $VMO^{1,\infty}$ graph property if it is locally the graph of a $VMO^{1,\infty}$-function. This is a particular type of VMO-smooth manifold with the local Lipschitz graph property, for which a G-atlas may be found so that $\Psi_j \in VMO^{1,\infty}(B_j)$ for all j. In this case, the Euclidean scalar product on $\mathbb{R}^{n+1}$ induces by restriction a VMO Riemannian metric on $\mathcal{M}$, thanks to (21) and the fact that $L^\infty(B_j) \cap VMO(B_j)$ is an algebra.

2. Sobolev spaces and vector fields

SobGD

Hereafter, we let $\mathcal{M}$ be a compact Lipschitz manifold of dimension n, assumed without loss of generality to be embedded in $\mathbb{R}^m$, and we pick a B-atlas $(U_j, \phi_j)_{1\leq j\leq N}$ in the sense of Section 1.4. We assume that $\mathcal{M}$ is endowed with a Riemannian metric $\Gamma_x : T_x\mathcal{M} \times T_x\mathcal{M} \to \mathbb{R}$, and σ indicates the associated volume measure. We recall the notation $L^p(\mathcal{M})$ or $L^p(\mathcal{M}, \mathbb{R}^k)$ for the corresponding Lebesgue spaces on $\mathcal{M}$.
A σ-measurable map $\mathcal{M} \to \mathbb{R}^m$ will be called a vector field on $\mathcal{M}$, and we identify vector fields that coincide a.e. A vector field X is said to be tangent if $X(x) \in T_x\mathcal{M}$ for σ-a.e. x. The subspace of tangent vector fields in $L^p(\mathcal{M}, \mathbb{R}^m)$ is denoted by $\mathcal{T}^p(\mathcal{M})$. It is closed because X is tangent if and only if, for each j, all $(n+1)\times(n+1)$ minors of the matrix $(D\phi_j^{-1}\circ\phi_j(x), X(x))$ vanish for σ-a.e. $x \in U_j$, while a convergent sequence in $L^p(\mathcal{M}, \mathbb{R}^m)$ has a subsequence converging pointwise a.e. We endow $\mathcal{T}^p(\mathcal{M})$ with the norm

$$\|G\|_{\mathcal{T}^p(\mathcal{M})} = \left(\int_{\mathcal{M}} \big(\Gamma_x(X,X)\big)^{p/2}\, d\sigma(x)\right)^{1/p}, \tag{26}$$ equivNT

where the integral is replaced by *essential sup* $(\Gamma(X,X))^{1/2}$ if $p = \infty$. By (8), the norm $\|.\|_{\mathcal{T}^p(\mathcal{M}}$ is equivalent to the restriction of $\|.\|_{L^p(\mathcal{M},\mathbb{R}^m)}$ to $\mathcal{T}^p(\mathcal{M})$.
For $1 \leq p < \infty$, the dual of $\mathcal{T}^p(\mathcal{M})$ is isometrically $\mathcal{T}^{p'}(\mathcal{M})$ under the pairing

$$(X,Y)_\Gamma = \int_{\mathcal{M}} \Gamma_x(X,Y) d\sigma(x), \qquad X \in \mathcal{T}^p(\mathcal{M}),\ Y \in \mathcal{T}^{p'}(\mathcal{M}). \tag{27}$$ pairingT

Indeed, applying the Schwarz inequality to Γ_x and Hölder's inequality, we get that $L : \mathcal{T}^{p'}(\mathcal{M}) \to (\mathcal{T}^p(\mathcal{M}))^*$ given by $L(Y)(X) = (X,Y)_\Gamma$, is a contractive linear map, and taking $X = (\Gamma(Y,Y))^{p'/2-1}Y$ shows that L is an isometry. To see it is surjective, observe that every $\Phi \in (\mathcal{T}^p(\mathcal{M}))^*$ induces in local coordinates a linear form on $L^p(B_j, \mathbb{R}^n)$ which can be represented as $f \mapsto \int_{B_j} f h_j dm_n$ for some $h_j \in L^{p'}(B_j, \mathbb{R}^n)$, by standard duality. Then, the vector field $H_j \in \mathcal{T}^{p'}(\mathcal{M})$ defined by $H_j = D\phi_j^{-1}((g_{i,k}^{-1})h_j/\sqrt{g^{(j)}})$ on U_j and 0 elsewhere satisfies $\Phi(X) = L(H_j)(X)$ for those $X \in \mathcal{T}^p(\mathcal{M})$ that vanish

off U_j. Letting (φ_j) be a Lipschitz partition of unity subordinated to (U_j), we find that $L(\sum_j \varphi_j H_j) = \Phi$. Now, since L is injective and surjective, it is an isomorphism by the open mapping theorem, as announced.
If $\Phi : \mathcal{M} \to \mathbb{R}$ is differentiable at some regular x, the derivative $D\Phi(x)$ belongs to $T_x^*\mathcal{M}$, thus it can be represented as $D\Phi(x)(X) = \Gamma_x(\nabla\Phi(x), X)$ for all $X \in T_x\mathcal{M}$ and some unique vector $\nabla\Phi(x) \in T_x\mathcal{M}$, called the gradient of Φ at x. It follows from (9) and the chain rule that, in the chart (U_j, ϕ_j), the local representative $v(y)$ of $\nabla\Phi(x)$ at $y = \phi_j(x)$ is related to the Euclidean gradient $\nabla(\Phi \circ \phi_j^{-1})(y)$ of $\Phi \circ \phi_j^{-1}$ at y by the formula:

$$v(y) = (g_{i,k})^{-1}(y)\nabla(\Phi \circ \phi_j^{-1})(y), \qquad D\phi_j^{-1}(y)v(y) = \nabla\Phi(x), \quad y = \phi_j(x). \tag{28}$$

corgrad

The Sobolev space $W^{1,\infty}(\mathcal{M})$ consists of real-valued Lipschitz functions on $\mathcal{M}$, endowed with the norm

$$\|\psi\|_{W^{1,\infty}(\mathcal{M})} = \|\psi\|_{L^\infty(\mathcal{M})} + \|\nabla\psi\|_{\mathcal{T}^\infty(\mathcal{M})}. \tag{29}$$

defSobinf

Note that indeed $\nabla\psi \in \mathcal{T}^\infty(\mathcal{M})$ when ψ is Lipschitz on $\mathcal{M}$, which is equivalent to say that $\psi \circ \phi_j^{-1}$ is Lipschitz on B_j for each j.
For $1 \leq p < \infty$, the Sobolev space $W^{1,p}(\mathcal{M})$ is the Banach space obtained as the completion of $W^{1,\infty}(\mathcal{M})$ with respect to the norm:

$$\|\psi\|_{W^{1,p}(\mathcal{M})} = \left(\|\psi\|^p_{L^p(\mathcal{M})} + \|\nabla\psi\|^p_{\mathcal{T}^p(\mathcal{M})}\right)^{1/p}. \tag{30}$$

defSob

Note that $\psi \in W^{1,p}(\mathcal{M})$ if and only if, for each chart (U_j, ϕ_j) the function $\psi \circ \phi_j^{-1}$ belongs to the Euclidean Sobolev space $W^{1,p}(B_j)$: using a Lipschitz partition of unity, this follows easily from (28), (10) and the fact that Lipschitz functions are dense in $W^{1,p}(B_j)$, see [1, Thm. 3.17]. Moreover $\sum_j \|\psi \circ \phi_j^{-1}\|_{W^{1,p}(B_j)}$ is a norm equivalent to $\|\psi\|_{W^{1,p}(\mathcal{M})}$. Passing to the limit with respect to a sequence (Φ_k) of Lipschitz functions in (28), we get that each $\psi \in W^{1,p}(\mathcal{M})$ has a well-defined gradient $\nabla\psi \in \mathcal{T}^p(\mathcal{M})$. Note that $\nabla\psi(x)$ needs not be a directional derivative of ψ in the strong sense if $p \leq n$ [51, Ch.VIII]. Instead, the gradient is characterized by the property that ψ is p-flat as a 0-form, with distributional exterior derivative $d\psi$ given by $d\psi(x)(X) = \Gamma_x(\nabla\psi(x), X)$ for $X \in T_x(\mathcal{M})$.
Proceeding in local coordinates, one easily gets a Sobolev embedding theorem for $W^{1,p}(\mathcal{M})$ from the well-known Euclidean version [1, thms. 4.12]. Namely, if $p > n$ then $W^{1,p}(\mathcal{M})$ embeds continuously in Hölder-continuous

functions on $\mathcal{M}$ with exponent $1-n/p$ (for the metric induced on $\mathcal{M}$ by $\mathbb{R}^m$ is compatible with the Euclidean metrics on the B_j), while for $1 \le p < n$ the embeding is into $L^{p^*}(\mathcal{M})$ with $p^* = np/(n-p)$, and $W^{1,n}(\mathcal{M})$ embeds in $L^\ell(\mathcal{M})$ for $\ell \in [1,\infty)$.
Likewise, we get an analog of the Rellich-Kondrachov theorem [1, thm. 6.3]: for $p \le n$ the embedding $W^{1,p}(\mathcal{M}) \to L^\ell(\mathcal{M})$ is compact when $\ell \in [1,p^*)$. Moreover, $W^{1,p}(\mathcal{M})$ is reflexive for $1 < p < \infty$, as it identifies with a closed subspace of $L^p(\mathcal{M}) \times L^p(\mathcal{M},\mathbb{R}^m)$ under the map $\psi \mapsto (\psi, \nabla\psi)$. With these results in mind, the argument given in [30, Lemma 3.8 & Prop. 3.9] for smooth $\mathcal{M}$ applies without change to give us:

$$\left\|\psi - \frac{1}{\sigma(\mathcal{M})}\int_{\mathcal{M}} \psi d\sigma\right\|_{L^{p^*}(\mathcal{M})} \le C\|\nabla\psi\|_{\mathcal{T}^p(\mathcal{M})}, \quad \psi \in W^{1,p}(\mathcal{M}), \quad 1 < p < n. \tag{31}$$

Poincareinegr

From (31) we easily obtain a Poincaré inequality for $1 < p < \infty$:

Poincarep **Lemma 2.1.** *For $1 < p < \infty$, there is a constant $C = C(\mathcal{M}, p)$ such that*

$$\left\|\psi - \frac{1}{\sigma(\mathcal{M})}\int_{\mathcal{M}} \psi d\sigma\right\|_{L^{p}(\mathcal{M})} \le C\|\nabla\psi\|_{\mathcal{T}^p(\mathcal{M})}, \qquad \psi \in W^{1,p}(\mathcal{M}). \tag{32}$$

Poincareineg

Proof. Since $p* > p$ and $\sigma(\mathcal{M})$ is finite, we get (32) from (31) and Hölder's inequality if $p < n$. Next, assume that $p \ge n$ and pick $\psi \in W^{1,p}(\mathcal{M})$. If we set $r = np/(n+p)$, then $1 < r < n \le p = r^*$ except when $n = p = 2$ (then $r = 1$). Save in this case, applying (31) with p replaced by r and observing from Hölder's inequality that $\|\nabla\psi\|_{\mathcal{T}^r(\mathcal{M})} \le C'\|\nabla\psi\|_{\mathcal{T}^p(\mathcal{M})}$, we get (32). To deal with the pending case $p = n = 2$, pick $q \in (1,2)$ so that $q^* > 2$ and write (31) with q instead of p. Using Hölder's inequality on both sides yields (32) in this case as well. □

We denote with $\mathcal{G}^p(\mathcal{M})$ the subspace of $\mathcal{T}^p(\mathcal{M})$ comprised of gradients of $W^{1,p}(\mathcal{M})$-functions. That is: $G \in \mathcal{G}^p(\mathcal{M})$ if and only if there is $f \in W^{1,p}(\mathcal{M})$ with $G = \nabla f$. Of course, f is defined up to an additive constant only.

gradfer **Lemma 2.2.** *For $1 < p < \infty$, the space $\mathcal{G}^p(\mathcal{M})$ is closed in $\mathcal{T}^p(\mathcal{M})$.*

Proof. Let $(G_k)_{k\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{G}^p(\mathcal{M})$ with $G_k = \nabla f_k$ for some $g_k \in W^{1,p}(\mathcal{M})$, normalized so that $\int_{\mathcal{M}} f_k d\sigma = 0$. Then, G_k converges to a limit Y in $\mathcal{T}^p(\mathcal{M})$, and we need to show there is $h \in W^{1,p}(\mathcal{M})$ with $\nabla h = Y$. By Lemma 2.1 f_k is bounded in $L^p(\mathcal{M})$, so we can extract a subsequence, again denoted by f_k, that converges weakly to some $h \in L^p(\mathcal{M})$. Writing (17) with $k = 0$, $\omega = f_k$ and $df_k(X) = \langle G_k, X\rangle$, we see upon passing to the limit for fixed j as $k \to \infty$ that $X \mapsto \langle Y, X\rangle$ is the distributional exterior derivative of h, as desired. $\square$

For $1 < p < \infty$ and $1/p + 1/p' = 1$, the dual $W^{-1,p'}(\mathcal{M})$ of $W^{1,p}(\mathcal{M})$ may be realized as the completion of $L^{p'}(\mathcal{M})$ for the norm

$$\|f\|_{W^{-1,p'}(\mathcal{M})} := \sup_{\|\psi\|_{W^{1,p}(\mathcal{M})=1}} \left| \int_{\mathcal{M}} f\psi \, d\sigma \right| ;$$

the proof goes as in the Euclidean case by the reflexivity of $W^{1,p}(\mathcal{M})$, see [1, Sec. 3.13].

Let now $\delta : \mathcal{T}^{p'}(\mathcal{M}) \to W^{-1,p'}(\mathcal{M})$ be the adjoint of the gradient operator $\nabla : W^{1,p}(\mathcal{M}) \to \mathcal{T}^p(\mathcal{M})$ mapping ψ to $\nabla\psi$. The operator δ is minus the divergence operator on $\mathcal{T}^{p'}(\mathcal{M})$. We denote by $\mathcal{D}^{p'}(\mathcal{M})$ the kernel of δ. By construction $\mathcal{D}^{p'}(\mathcal{M})$ is a closed subspace of $\mathcal{T}^{p'}(\mathcal{M})$, the elements of which are said to be divergence-free.

It follows at once from the definition that $\mathcal{D}^{p'}(\mathcal{M})$ is the space orthogonal to $\mathcal{G}^p(\mathcal{M})$ under the pairing (27). It is then a consequence of the Hahn-Banach theorem [16, Thm. 7.1] that, for $X \in \mathcal{T}^p(\mathcal{M})$, the norm of the coset $\dot{X}$ in the quotient space $\mathcal{T}^p(\mathcal{M})/\mathcal{G}^p(\mathcal{M})$ is the norm of the functional $D \mapsto (X, D)_\Gamma$ on $\mathcal{D}^{p'}(\mathcal{M})$. In particular, if $X \in \mathcal{T}^p(\mathcal{M})$ is orthogonal to $\mathcal{D}^{p'}(\mathcal{M})$ *via* (27), it holds that $X \in \mathcal{G}^p(\mathcal{M})$.

By (28) and the divergence formula, we get on using a Lipschitz-smooth partition of unity subordinated to $(U_j)_{1\le j\le N}$ that a tangent vector field $V \in \mathcal{T}^{p'}(\mathcal{M})$ lies in $\mathcal{D}^{p'}(\mathcal{M})$ if and only if, for every j, its local representative v in the chart $\phi_j : U_j \to B_j$ is a distributional solution in $L^{p'}(B_j, \mathbb{R}^n)$ to the equation:

$$\mathrm{div}(\sqrt{g^{(j)}}\, v) = 0, \qquad D\phi_j^{-1}(y)(v(y)) = V(x), \quad y = \phi_j(x), \tag{33}$$ locdf

where $\mathrm{div} = \sum_{i=1}^n \frac{\partial}{\partial y_i}$ is the Euclidean divergence operator.

3. The Helmholtz-Hodge decomposition

sec:HH

A Helmholtz-Hodge decomposition is a direct sum decomposition, in a given regularity class on a manifold, of all tangent vector fields into gradient vector fields and divergence-free vector fields, see [8] for some motivation and history.

In Euclidean space, it was first established by Helmholtz for smoothly decaying vector fields in $\mathbb{R}^3$, and later carried over to vector fields in $L^p(\mathbb{R}^n, \mathbb{R}^n)$ for $1 < p < \infty$, see [45, 35]. That is, if $\mathcal{D}^p(\mathbb{R}^n)$ denotes the space of those $X \in L^p(\mathbb{R}^n, \mathbb{R}^n)$ such that $\operatorname{div} X = 0$ in the distributional sense, and if $\mathcal{G}^p(\mathbb{R}^n) \subset L^p(\mathbb{R}^n, \mathbb{R}^n)$ designates the subspace of gradients of functions in $\dot{W}^{1,p}(\mathbb{R}^n)$, then there is a topological sum:

$$L^p(\mathbb{R}^n, \mathbb{R}^n) = \mathcal{G}^p(\mathbb{R}^n) \oplus \mathcal{D}^p(\mathbb{R}^n), \qquad 1 < p < \infty. \tag{34}$$ HeHoRk

The sum is indeed direct, because no nonconstant harmonic function on $\mathbb{R}^n$ has L^p derivatives. On a bounded C^1-smooth domain $\Omega \subset \mathbb{R}^n$, a similar decomposition holds, namely:

$$L^p(\Omega, \mathbb{R}^n) = \mathcal{G}_0^p(\Omega) \oplus \mathcal{D}^p(\Omega), \qquad 1 < p < \infty, \tag{35}$$ HeHoRk1

where $\mathcal{D}^p(\Omega)$ denotes the space of divergence-free vector fields in $L^p(\Omega, \mathbb{R}^n)$, in the distributional sense, and $\mathcal{G}_0^p(\Omega) \subset L^p(\mathbb{R}^n, \mathbb{R}^n)$ is the space of gradients of functions in $W_0^{1,p}(\Omega)$; when $\partial\Omega$ merely has the Lipschitz graph property, (35) still holds, at least for $3/2 - \varepsilon < p < 3 + \varepsilon$ where $\varepsilon > 0$ depends on Ω, see [20, Thm. 11.2].

On a bounded C^∞-smooth domain, decomposition (35) is actually a byproduct of Hodge theory, asserting that a k-form on Ω is uniquely the sum of three terms: (i) the exterior derivative of a $(k-1)$-form with vanishing tangential component on the boundary $\partial\Omega$, (ii) the co-differential of a $(k+1)$-form with vanishing normal component on $\partial\Omega$, (iii) a harmonic k-form [35, Thm. 10.5.1]. Here, the co-differential is the adjoint to the exterior derivative for the Hodge scalar product, and when applied to a 1-form it yields the divergence of the vector field representing the form *via* the Euclidean scalar product. As co-differentiating twice yields the zero map and since harmonic forms have zero co-differential, we see that (35) indeed follows from the Hodge decomposition at $k = 1$. Note that the Helmholtz-Hodge decomposition is less precise, as it merges (ii) and (iii) into a single, divergence-free term.

Similar considerations apply on a C^∞-smooth, compact and oriented Riemannian manifold $\mathcal{M}$, where classical Hodge theory was extended to forms of L^p-class in [49]. When applied to 1-forms, it entails again that

$$\mathcal{T}^p(\mathcal{M}) = \mathcal{G}^p(\mathcal{M}) \oplus \mathcal{D}^p(\mathcal{M}), \qquad 1 < p < \infty, \tag{36}$$ HeHodecMp

see [39, 3] for generalizations to the noncompact case.

On a closed oriented manifold which is merely Lipschitz-smooth, Hodge theory was carried over to forms of L^2-class in [53], as a tool to develop index theory for signature operators. This far reaching generalization implies of course the existence of a Helmholtz-Hodge decomposition in $\mathcal{T}^2(\mathcal{M})$, but the latter is obvious anyway for it reduces to the decomposition of the Hilbert space $\mathcal{T}^2(\mathcal{M})$ as the sum of the closed subspace $\mathcal{G}^2(\mathcal{M})$ and its orthogonal complement. When $p \neq 2$, Hodge theory for forms of L^p -class on a Lipschitz manifold has apparently not been adressed so far.
Our goal in this section is to establish a Helmoltz-Hodge decomposition for tangent vector fields of L^p-class on a compact (not necessarily oriented) Lipschitz Riemannian manifold $\mathcal{M}$, at least when p is close enough to 2, and in fact for all $p \in (1, \infty)$ if, in addition, the metric tensor in local coordinates is of VMO-class. This result will be used in Section 6 to establish the Hardy-Hodge decomposition on Lipschitz hypersurfaces with the local graph property. We shall need a non-Euclidean version of (35), namely:

$$L^p(\Omega, \mathbb{R}^n) = \mathcal{G}_0^p(\Omega) \oplus \mathcal{D}_A^p(\Omega), \tag{37}$$

HeHoRkA

where $\Omega \subset \mathbb{R}^n$ is a bounded open set whose boundary is locally a Lipschitz graph, $\mathcal{G}_0^p(\Omega)$ is the space of gradients of functions in $W_0^{1,p}(\Omega)$, and

$$\mathcal{D}_A^p(\Omega) = \{D \in L^p(\Omega, \mathbb{R}^n) : \ \mathrm{div}(AD) = 0\} \tag{38}$$

DA

in the distributional sense on Ω, for some function $A : \Omega \to \mathbb{R}^{n\times n}$, valued in the set of positive-symmetric matrices, satisfying a strict ellipticity condition:

$$0 < c_1 I_n \leq A \leq C_1 I_n \qquad \text{a.e. on } \Omega. \tag{39}$$

ellipticcond

Clearly, (37) holds as a topological sum if and only if the equation

$$\mathrm{div}(A\nabla u) = \mathrm{div}(AF) \tag{40}$$

divVMO

can be solved uniquely with respect to $u \in W_0^{1,p}(\Omega)$ for each $F \in L^p(\Omega)$, because then the natural map $\mathcal{G}_0^p(\Omega) \oplus \mathcal{D}_A^p(\Omega) \to L^p(\Omega, \mathbb{R}^n)$ is surjective and injective, hence a homeomorphism by the open mapping theorem. This is actually the case when $p \in (q', q)$, for some $q > 2$ depending on Ω and the constants in (39), see [44, Cor. 4]. If in addition Ω is C^1-smooth and the coefficients of A lie in $VMO(\Omega)$, then we may take $q = \infty$ [17]. Note that, by reason of homogeneity, the exponent q as well as the norms of the projections in (37) remain unchanged under a dilation of Ω and the corresponding dilation of the argument of A. In particular, if $\Omega \subset \mathbb{R}^k$ is a ball, these are uniformly bounded in terms of k and the constants in (39), independently of the radius of that ball.

HHMG **Theorem 3.1** (Helmholtz-Hodge decomposition)**.** *Let $\mathcal{M}$ be a compact Lipschitz manifold endowed with a Riemannian metric. There exists $\varepsilon \in (0, 1/2]$ such that, for $|1/2 - 1/p| < \varepsilon$, the following topological sum holds:*

$$\mathcal{T}^p(\mathcal{M}) = \mathcal{G}^p(\mathcal{M}) \oplus \mathcal{D}^p(\mathcal{M}). \tag{41}$$ HeHodecM

If, in addition, $\mathcal{M}$ is a compact, VMO-smooth manifold equipped with a VMO Riemannian metric, then (41) *holds for* $1 < p < \infty$.

Proof. Let $(U_j, \phi_j)_{1 \leq j \leq N}$ be a B-atlas such that $B_j := \phi_j(U_i)$ is an open ball in $\mathbb{R}^n$ for each j. Define a positive matrix-valued function A_j on B_j by $A_j := \sqrt{g}(g_{i,k})^{-1}$; here, for simplicity, we do not show the dependence of g, $g_{i,k}$ does not show the dependence on j. Observe from (10) that A_j satisfies (39) on B_j, where c_1, C_1 can be made independent of j. Thus, by the discussion after (39), there is $q > 2$ such that (37) holds for all $p \in (q', q)$ when Ω is set to B_j and A is set to A_j, for every $j \in \{1, \cdots, N\}$. We shall prove that (41) holds if $p \in (q', q)$. The case $p = 2$ is trivial, as pointed out already, therefore we need only consider the cases where $p \in (2, q)$ and $p \in (q', 2)$.

We first show that when $p \in (2, q)$, each member of $\mathcal{T}^p(\mathcal{M})$ can uniquely be written in the form $G + D$, with $G \in \mathcal{G}^p(\mathcal{M})$ and $D \in \mathcal{D}^p(\mathcal{M})$. By the open mapping theorem, this will establish (41) for such p, because $\mathcal{M}$ is compact so that $\mathcal{T}^p(\mathcal{M}) \subset \mathcal{T}^2(\mathcal{M})$, whence the sum $\mathcal{G}^p(\mathcal{M}) + \mathcal{D}^p(\mathcal{M})$ is direct as it is direct for $p = 2$. To proceed, fix $F \in \mathcal{T}^p(\mathcal{M})$. *A fortiori* $F \in \mathcal{T}^2(\mathcal{M})$, hence we can write $F = G + D$ with $D \in \mathcal{D}^2(\mathcal{M})$ and $G \in \mathcal{G}^2(\mathcal{M})$, say $G = \nabla f$ where $f \in W^{1,2}(\mathcal{M})$. Let (φ_j) a Lipschitz partition of unity subordinated to the (U_j). Set $h = f \circ \phi_j^{-1}$ and $\eta_j = \varphi_j \circ \phi_j^{-1}$, so that $h \in W^{1,2}(B_j)$ and $\eta_j \in W^{1,\infty}(B_j)$. Let F_j, D_j be the vector fields on B_j which are local representatives of F, D; *i.e.*, $F_{|U_j} = D\phi_j^{-1}(F_j)$ and $D_{|U_j} = D\phi_j^{-1}(D_j)$. Since ϕ_j^{-1} is bi-Lipschitz. we have that $F_j \in L^p(B_j, \mathbb{R}^n)$ and $D_j \in L^2(B_j, \mathbb{R}^n)$, by (3). Consider the two vector fields on B_j given by $W = \sqrt{g}F_j$ and $X = (g_{i,k})D_j$ which lie in $L^p(B_j, \mathbb{R}^n)$ and $L^2(B_j, \mathbb{R}^n)$ respectively, thanks to (10). In local coordinates, we see from (28) that the relation $\varphi_j F = \varphi_j G + \varphi_j D$ is equivalent to

$$\eta_j A_j^{-1} W = \eta_j \nabla h + \eta_j X \qquad \text{on } B_j, \tag{42}$$ decs

where A_j was defined above. It follows from (33) that $\mathrm{div}(A_jX) = 0$, in the distributional sense on B_j, hence $\mathrm{div}(A_j\eta_jX) = \langle \nabla\eta_j, A_jX\rangle$ lies in $L^2(B_j)$ because $\nabla\eta_j \in L^\infty(B_j, \mathbb{R}^n)$ and A_j satisfies (39). Thus, by elliptic regularity on the smooth domain B_j (see *e.g.* [26, Thm. 9.15]), there is $u \in W_0^{2,2}(B_j)$ such that $\Delta u = \mathrm{div}(A_j\eta_jX)$, where $\Delta = \sum_{k=1}^n \partial^2_{y_k}$ is the Euclidean Laplacian. Setting $Z = \eta_jX - A_j^{-1}\nabla u$, we get that $Z \in L^2(B_j, \mathbb{R}^n)$ with $\mathrm{div}A_jZ = 0$, and we can rewrite (42) as

$$\eta_jA_j^{-1}W + h\nabla\eta_j - A_j^{-1}\nabla u \,=\, \nabla(h\eta_j) + Z \qquad \text{on } B_j, \tag{43}$$

decs1

Note that $h\eta_j \in W_0^{1,2}(B_j)$ since η_j is compactly supported in B_j, therefore (43) expresses the unique decomposition of the left-hand side in $\mathcal{G}_0^2 \oplus \mathcal{D}^2_{A_j}$. *We claim* that $\varphi_jf \in W^{1,\alpha}(\mathcal{M})$ whenever the left-hand side of (43) lies in $L^\alpha(B_j, \mathbb{R}^n)$ with $\alpha \in (2, p]$. Indeed, by (37), this left-hand side can then be written uniquely as $\nabla h_1 + D_1$ where $h_1 \in W_0^{1,\alpha}(B_j)$ and $D_1 \in L^\alpha(B_j, \mathbb{R}^n)$, with $\mathrm{div}\,(A_jD_1) = 0$. As $L^\alpha(B_j, \mathbb{R}^n) \subset L^2(B_j, \mathbb{R}^n)$ because B_j is bounded, we thus get two decompositions of the left-hand side in $\mathcal{G}_0^2 \oplus \mathcal{D}^2_{A_j}$ and they must coincide: $\nabla(h\eta_j) = \nabla h_1$. Consequently $\nabla(h\eta_j) \in L^\alpha(B_j, \mathbb{R}^n)$, hence $h\eta_j = (\varphi_jf) \circ \phi_j^{-1}$ belongs to $W^{1,\alpha}(B_j)$ by the Poincaré inequality. Since φ_jf is supported in U_j it follows that $\varphi_jf \in W^{1,\alpha}(\mathcal{M})$, *thereby proving the claim.* We now use the claim inductively to prove that $f \in W^{1,p}(\mathcal{M})$.
If $n = 2$, then $h \in L^\ell(B_j)$ and $\nabla u \in L^\ell(B_j, \mathbb{R}^2)$ for every $\ell \in [1, \infty)$, by the Sobolev embedding theorem. Hence, the left-hand side of (43) lies in $L^p(B_j, \mathbb{R}^2)$, since $\nabla\eta_j$ is bounded and A_j satisfies (39). Thus, we get from the claim that $\varphi_jf \in W^{1,p}(\mathcal{M})$, and since this holds for each $j \in \{1, \cdots, N\}$ we get that $f = \sum_j(\varphi_jf) \in W^{1,p}(\mathcal{M})$, as desired.
Suppose next that $n \geq 3$ and set $p_1 := 2n/(n-2)$. By the Sobolev embedding theorem, we get that $h \in L^{p_1}(B_j)$ and $\nabla u \in L^{p_1}(B_j, \mathbb{R}^n)$, so the left-hand side of (43) belongs to $L^{p_1}(B_j, \mathbb{R}^n)$ and the claim implies that $\varphi_jf \in W^{1,\min\{p,p_1\}}(\mathcal{M})$ for each j. If $p_1 \geq p$ we are done, otherwise $f \in W^{1,p_1}(\mathcal{M})$, implying that $G = \nabla f$ and $D = F - G$ belong to $\mathcal{T}^{p_1}(\mathcal{M})$. Consequently $h \in W^{1,p_1}(B_j)$ and $X \in L^{p_1}(B_j, \mathbb{R}^n)$ for each j, and the latter implies that $\mathrm{div}(A_j\eta_jX) = \langle \nabla\eta_j, A_jX\rangle \in L^{p_1}(B_j)$, whence $u \in W_0^{2,p_1}(B_j)$ by elliptic regularity on B_j. If $p_1 > n$ then h and ∇u are bounded on B_j, and if $p_1 = n$ they lie respectively in $L^\ell(B_j)$ and $L^\ell(B_j, \mathbb{R}^n)$ for every $\ell \in [1, \infty)$, by the Sobolev embedding theorem again. So, if $p_1 \geq n$ then the left-hand side of (43) *a fortiori* lies in $L^p(B_j, \mathbb{R}^n)$, and the claim implies that $\varphi_jf \in W^{1,p}(\mathcal{M})$, implying since this holds for every j that $f \in W^{1,p}(\mathcal{M})$, as desired. If on the

contrary $p_1 < n$, we set $p_2 := np_1/(n-p_1)$ and repeat the previous argument replacing p_1 by p_2. As the sequence $p_{k+1} = np_k/(n-p_k)$ will overshoot n or p after finitely many steps, this reasoning eventually leads us to the conclusion that $f \in W^{1,p}(\mathcal{M})$, so that G and $D = F - G$ lie in $\mathcal{T}^p(\mathcal{M})$, as wanted.
Having proven (41) for $p \in (2, q)$, we get it in the range of exponents $(q', 2)$ by duality, because (41) implies that every linear form on $\mathcal{T}^p(\mathcal{M})$ is uniquely the sum of a form vanishing on $\mathcal{G}^p(\mathcal{M})$ (*i.e.* a member of $\mathcal{D}^{p'}(\mathcal{M})$) and a form vanishing on $\mathcal{D}^p(\mathcal{M})$ (*i.e.* a member of $\mathcal{G}^{p'}(\mathcal{M})$).
Finally, if $\mathcal{M}$ is VMO-smooth with VMO Riemannian metric, we may pick $\cup_j(U_j, \phi_j)_{1\leq j\leq N}$ to be a VMO-smooth B-atlas such that $(g^{(j)}_{i,k})$ belongs to $VMO(B_j, \mathbb{R}^{n\times n})$ for each j, where $B_j := \phi_j(U_j)$ is a ball. In view of (10), Lemma 7.2 implies that $\sqrt{g^{(j)}}(g^{(j)}_{i,k})^{-1}$ lies in $VMO(B_j, \mathbb{R}^{n\times n})$. With the previous notation, this means that A_j belongs to $VMO(B_j, \mathbb{R}^n)$ for all j, therefore we may take $q = \infty$ in the above proof to conclude that (41) is valid for $1 < p < \infty$. □

4. Some properties of harmonic functions on Lipschitz domains

hfl

Let $\mathcal{M} \subset \mathbb{R}^{n+1}$ be a compact and connected hypersurface with the local Lipschitz graph property. We endow $\mathcal{M}$ with the Riemannian metric induced by Euclidean scalar product on $\mathbb{R}^{n+1}$, and denote by σ the corresponding volume measure (*i.e.* $\sigma = \mathcal{H}^n\lfloor\mathcal{M}$). We write Ω^+ and Ω^- for the interior and exterior components of $\mathbb{R}^n \setminus \mathcal{M}$, respectively. Lipschitz domains referred to in the title of this section are those of the form $\Omega^\pm$, for some $\mathcal{M}$ as above.
To $\xi \in \mathcal{M}$ and $\alpha > 1$, we associate two nontangential regions of approach to ξ, one from Ω^+ and one from Ω^-, given by

$$R^{\Omega^\pm}_\alpha(\xi) = \{x \in \Omega^\pm :\ |x - \xi| < \alpha\, \mathrm{d}(x, \mathcal{M})\}. \tag{44}$$ rNT

The nontangential maximal function (associated with α) of $h : \Omega^\pm \to \mathbb{R}^k$ is $\mathcal{N}^{\Omega^\pm}_\alpha h : \mathcal{M} \to [0, +\infty]$ defined by

$$\mathcal{N}^{\Omega^\pm}_\alpha h(\xi) = \sup_{x\in R^{\Omega^\pm}_\alpha(\xi)} |h(x)|, \qquad \xi \in \mathcal{M}. \tag{45}$$ defNT

We say that h converges nontangentially to a at $\xi \in \mathcal{M}$ from $\Omega^\pm$, abbreviated as $h \overset{n.t.\Omega^\pm}{\to} a$ at ξ, if for each $\alpha > 1$ we have that

$$\lim_{x\to\xi,\, x\in R^{\Omega^\pm}_\alpha(\xi)} h(x) = a. \tag{46}$$ defntc

It can happen that $\xi \notin \overline{R_\alpha^{\Omega^\pm}(\xi)}$, and in fact $R_\alpha^{\Omega^\pm}(\xi)$ may even be empty for some ξ and α, but since the measure-theoretic boundary of $\Omega^\pm$ is $\mathcal{M}$ (see Section 1.6) such ξ form a set of σ-measure zero [33, Prop. 2.3.2]. Hence (46) makes sense, at least for σ-a.e ξ. The property that $\mathcal{N}_\alpha^{\Omega^\pm} h \in L^p(\mathcal{M})$ holds for every $\alpha > 1$ if it holds for one of them. This fact, which depends on the Ahlfors-David regularity of $\mathcal{M}$ (see (14)), is stated in [33, Prop. 2.12] for $0 < p < \infty$, but the proof works for $p = \infty$ as well. More precisely, if for $f : \mathcal{M} \to \mathbb{R}$ we let $f^* : \mathbb{R}^+ \to \mathbb{R}^+$ indicate its distribution function:

$$f^*(\lambda) = \sigma\left(\{\xi \in \mathcal{M} : \, |f(\xi)| > \lambda\}\right), \qquad \lambda \geq 0, \tag{47}$$ distfun

then [33, Eqn. 2.1.19] shows that $(\mathcal{N}_\alpha^{\Omega^\pm} h)^*$ and $(\mathcal{N}_\beta^{\Omega^\pm} h)^*$ are equivalent for any $\alpha, \beta > 1$, up to a multiplicative constant depending on α, β and the constants in (14). Thus, as the L^p-norm is computed from the distribution function in an increasing manner, a minor variation of [33, Prop. 2.1.2] is:

compfmd **Lemma 4.1.** *Let $1 < \alpha < \beta$ and $h : \Omega^\pm \to \mathbb{R}^k$. To each $p \in [1, \infty]$, there is a constant C_1 depending on α, β, and Ω such that*

$$\|\mathcal{N}_\alpha^{\Omega^\pm} h\|_{L^p(\mathcal{M})} \leq \|\mathcal{N}_\beta^{\Omega^\pm} h\|_{L^p(\mathcal{M})} \leq C_1 \|\mathcal{N}_\alpha^{\Omega^\pm} h\|_{L^p(\mathcal{M})}. \tag{48}$$ inegcentcomp

The previous notions of nontangential limit and maximal function are intrinsic, in that the definition of $R_\alpha^{\Omega^\pm}(\xi)$ in (44) does not depend on a particular atlas for $\mathcal{M}$. However, seminal works on the Dirichlet and regularity problems, like [34, 11, 10, 12, 54], use another notion of nontangential approach, taken locally in a chart of some G-atlas, on the "natural" cones $C_{\theta, v_j^\pm}(\xi)$ defined in Section 1.6. Estimates obtained in each chart can then be glued together using a *regular family of cones* to obtain global estimates, as in [12, 54, 2]. With the notation set up in Section 1.6, a regular family of cones for $\Omega^\pm$ is a map associating to every $\xi \in \mathcal{M}$ a truncated cone $C_{\theta, \pm z(\xi)}(\xi) \subset \Omega^\pm$, where θ is independent of ξ and $z : \mathcal{M} \to \mathbb{S}^n$ is a continuous function into the n-dimensional unit sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, in such a way that for some G-atlas (U_j, ϕ_j), some cover (V_j) of $\mathcal{M}$ with $\overline{V}_j \subset U_j$ for each j, every $\xi \in \mathcal{M}$ and all j for which $\xi \in V_j$, the cone $C_{\theta, \pm z(\xi)}(\xi)$ cut off to suitable length independent of ξ and j contains a natural cone $C_{\theta_1, \pm v_j^\pm}(\xi)$ relative to U_j, V_j, and is contained in another natural cone $C_{\theta_2, \pm v_j^\pm}(\xi)$, with $\theta_1 < \theta_2$ independent of ξ:

$$\overline{C_{\theta_1, v_j^\mp}(\xi)} \setminus \{\xi\} \subset C_{\theta, \pm z(\xi)}(\xi) \subset \overline{C_{\theta, \pm z(\xi)}(\xi)} \subset C_{\theta_2, v_j^\mp}(\xi) \subset \mathcal{C}_j \cap \Omega^\pm. \tag{49}$$ incC

The existence of a regular family of cones is folklore, but it seems very hard to see a proof[1] of this technical result pertaining more to convex geometry than to analysis. We provide such a proof in Lemma 7.8, in which the G-atlas (U_j, ϕ_j) associated to the regular family of cones is such that $B_j = \phi_j(U_j)$ is a ball while V_j is of the form $\phi_j^{-1}(\mu B_j)$, where $\mu \in (0,1)$ can be made arbitrary small with respect to the length of $\mathcal{C}_j$. This last fact is important to glue together certain estimates from works like [34, 10], because if μ is small enough with respect to the length of $\mathcal{C}_j$ then the cordinate cylinders over V_j have starlike intersection with $\Omega^\pm$ about points on the axis of $\mathcal{C}_j$, close to the base.

Associated to a regular family of cones $\{C_{\theta,\pm z}\}$ is a nontangential maximal operator $N^\pm_{\theta,z}$, defined for $h : \Omega^\pm \to \mathbb{R}^k$ by $N^\pm_{\theta,z}h(\xi) = \sup_{x\in C_{\theta,\pm z}(\xi)} |h(x)|$. We have thus two nontangential maximal functions for h, namely $N^\pm_{\theta,z}h$ and $\mathcal{N}^{\Omega^\pm}_\alpha h$, and the question arises of how they compare. It follows easily from (19) that $C_{\theta,\pm z(\xi)}(\xi) \subset R^{\Omega^\pm}_{1/\sin\epsilon}(\xi)$, therefore $\|N^\pm_{\theta,z}h\|_{L^p(\mathcal{M})} \leq c\|\mathcal{N}^{\Omega^\pm}_\alpha h\|_{L^p(\mathcal{M})}$ for $1 \leq p \leq \infty$ with $c = c(\alpha, \epsilon, \Omega)$, by Lemma 4.1. That a reverse inequality holds to the effect that $\|N^\pm_{\theta,z}h\|_{L^p(\mathcal{M})}$ and $\|\mathcal{N}^{\Omega^\pm}_\alpha h\|_{L^p(\mathcal{M})}$ are in fact equivalent, at least when h is harmonic, is tacitly implied in several works; see, *e.g.* [13]. Because the proof is nontrivial and we could not find a discussion in the literature, we provide an argument through Lemmas 7.9, 7.10. With these facts at our disposal, we freely recast estimates established in terms of $\|N^\pm_{\theta,z}h\|_{L^p(\mathcal{M})}$ as estimates in terms of $\|\mathcal{N}^{\Omega^\pm}_\alpha h\|_{L^p(\mathcal{M})}$, and *vice-versa*. We also get in the same stroke that such estimates do not depend, up to a multiplicative constant, of the regular family of cones we choose.

Next, recall for $z \in \Omega^+$ the harmonic measure ω^+_z, which is the Borel probability measure on $\mathcal{M}$ such that $\int \varphi d\omega^+_z = u^+_\varphi(z)$ where, for each continuous function $\varphi : \mathcal{M} \to \mathbb{R}$, we let u^+_φ be the solution to the Dirichlet problem in Ω^+ with data φ; that is, u^+_φ is harmonic in Ω^+ and continuous on $\mathcal{M} \cup \Omega^+$ with $(u^+_\varphi)_{|\mathcal{M}} = \varphi$; see, *e.g.* [4, Thms. 6.3.8 & 6.4.1 & 6.6.4]. Note that u^+_φ uniquely exists because Ω^+ is regular for the Dirichlet problem, since it satisfies the exterior cone condition [4, Thm. 6.6.15]. That the latter is fulfilled follows at once from the local Lipschitz graph property.

For $z \in \Omega^-$, we define ω^-_z to be the positive Borel measure on $\mathcal{M}$ such that $\int \varphi d\omega^-_z = u^-_\varphi(z)$ where, for a continuous function $\varphi : \mathcal{M} \to \mathbb{R}$, we let u^-_φ be

[1]Works we know either take this fact for granted or quote unpublished material, or else mention references in which we could not locate the result.

the solution to the Dirichlet problem in $\Omega^-\cup\{\infty\}$ with data φ; that is, u_φ^- is harmonic in $\Omega^-\cup\{\infty\}$ and continuous on $\mathcal{M}\cup\Omega^-\cup\{\infty\}$ with $(u_\varphi^-)_{|\mathcal{M}}=\varphi$. Here, recall since $n+1\geq 3$ that a harmonic function u on Ω^- is harmonic at infinity if $\lim_{|z|\to\infty}|u(z)|=0$ [6, Thm. 4.8], and that u_φ^- uniquely exists because $\{\infty\}$ is a regular point of $\partial^\infty\Omega^-:=\mathcal{M}\cup\{\infty\}$ [4, Thm. 6.7.1]. However, ω_z^- is not a probability measure, as it is the restriction to $\mathcal{M}$ of harmonic measure on Ω^- if $z\in\Omega^-$, while ω_∞^- is the zero measure.
The following construction is useful to relate ω_z^- to harmonic measure of a bounded Lipschitz domain: suppose without loss of generality that $0\in\Omega^+$ and consider the inversion $\mathcal{I}(x)=x/|x|^2$. It is a smooth conformal involution of the "sphere" $\mathbb{R}^{n+1}\cup\{\infty\}$, mapping Ω^- onto a bounded domain Ω_1, and the Kelvin transform $K[u](x):=|x|^{1-n}u(\mathcal{I}(x))$ establishes an involutive one-to-one correspondence between harmonic functions u on $\Omega^-\cup\{\infty\}$ and harmonic functions on Ω_1 [6, Ch. 4]. For $x_0\in\mathcal{M}$ and $x\in B(x_0,\rho)\subset\Omega^-$ with $\rho>0$ small enough, the image under $\mathcal{I}$ of a cone $C_{\theta,v}(x)\subset\Omega^-$, truncated to sufficiently small length r, contains a cone $C_{\theta_1,D\mathcal{I}(x_0)(v)}(\mathcal{I}(x))\subset\Omega_1$ truncated to sufficiently small length r_1, where $\theta_1,r_1>0$ are independent of x, by the smoothness and conformality of $\mathcal{I}$. Thus, Ω_1 has the uniform cone property since Ω^- does, implying that its boundary $\partial\Omega_1$ is locally a Lipschitz graph. Let $\omega_x^{\Omega_1}$ indicate harmonic measure on $\partial\Omega_1$ at $x\in\Omega_1$, and put $u_\Phi^{\Omega_1}$ for the solution to the Dirichlet problem in Ω_1 with continuous boundary values $\Phi:\partial\Omega_1\to\mathbb{R}$; that is: $u_\Phi^{\Omega_1}(x_1)=\int\Phi d\omega_{x_1}^{\Omega_1}$ for $x_1\in\Omega_1$. For every continuous $\varphi:\mathcal{M}\to\mathbb{R}$, it is readily checked that $u_\varphi^-=K[u^{\Omega_1}_{|\mathrm{Id}|^{1-n}\varphi\circ\mathcal{I}}]$, as both sides solve the classical Dirichlet problem on Ω^- with boundary values φ on $\mathcal{M}$ and 0 at infinity; here, Id denotes the identity map on $\partial\Omega_1$. Hence, we get for $x\in\Omega^-$ that $\int_{\mathcal{M}}\varphi(\xi)d\omega_x^-(\xi)=|x|^{1-n}\int_{\partial\Omega_1}|\zeta|^{1-n}\varphi\circ\mathcal{I}(\zeta)d\omega^{\Omega_1}_{\mathcal{I}(x)}$, which yields:

$$d\omega_x^-(\xi)=\frac{|x|^{1-n}}{|\xi|^{1-n}}d\mathcal{I}^*(\omega^{\Omega_1}_{\mathcal{I}(x)}),\qquad x\in\Omega^-,\quad \xi\in\mathcal{M},\tag{50}$$ CVFHM

where $\mathcal{I}^*(\omega^{\Omega_1}_{\mathcal{I}(x)})$ is the image of $\omega^{\Omega_1}_{\mathcal{I}(x)}$ under $\mathcal{I}$; *i.e.*, $\mathcal{I}^*(\omega_{\mathcal{I}(x)})(E)=\omega^{\Omega_1}_{\mathcal{I}(x)}(\mathcal{I}(E))$ for every Borel set $E\subset\mathcal{M}$.
Note that $z\mapsto\omega_z^\pm(E)$ is harmonic in $\Omega^\pm$, and one can see from Harnack's inequalities (see *e.g.* [4, Thm. 1.4.1]) that for each $z_0\in\Omega^\pm$ and any compact neighborhood K of z_0 in $\Omega^\pm$, there is a constant $C=C(\mathcal{M},K)$ with $\omega_z^\pm(E)/C\leq\omega_{z_0}^\pm(E)\leq C\omega_z^\pm(E)$ for all $z\in K$. Hence, $\omega_{z_1}^\pm$ and $\omega_{z_2}^\pm$ are mutually absolutely continuous for any $z_1,z_2\in\Omega^\pm$, with bounded Radon-Nykodim derivative of each one with respect to the other. In particular, one can speak

of a subset of $\mathcal{M}$ of harmonic measure zero. Also, it is worth observing from (50) and the Harnack inequalities that $d\omega_e := \lim_{|x|\to\infty} |x|^{n-1} d\omega_x^-$ exists in the strong sense as a positive measure on $\mathcal{M}$, with $d\omega_e(\xi) = |\xi|^{n-1} d\mathcal{I}^*(\omega_0^{\Omega_1})(\xi)$. In fact, ω_e is the so-called Newtonian equilibrium measure on $\Omega^+ \cup \mathcal{M}$, compare [32, Ch. IV, Sec. 5, §20].

The following fundamental fact regarding harmonic measure on Lipschitz domains was proven in [10]:

HMDa **Lemma 4.2.** *For $z \in \Omega^\pm$, the measures $\omega_z^\pm$ and σ are mutually absolutely continuous. Moreover, the Radon-Nykodim derivative $h_z^\pm := d\omega_z/d\sigma$ lies in $L^2(\mathcal{M})$, locally uniformly with respect to $z \in \Omega^\pm$.*

For the bounded domain Ω^+, Lemma 4.2 quickly follows from [10, Cor. to Thm. 3] and the observation that $h_z^+/C \le h_{z_0}^+ \le C h_z^+$ for z in a compact neighborhood of z_0, by previous remarks on harmonic measure. In view of (50), assuming without loss of generality that $0 \in \Omega^+$, the case of Ω^- follows by applying what we just said to the bounded domain $\Omega_1 = \mathcal{I}(\Omega^+)$, while observing that the inverse image of $\mathcal{H}^n \lfloor \partial\Omega_1$ under $\mathcal{I}$ has differential $|\xi|^{-2n} d\sigma(\xi)$ because, for $x \in \mathbb{R}^{n+1} \setminus \{0\}$, the derivative $D\mathcal{I}(x)$ is a similarity transformation with ratio $|x|^{-2}$, see [6, Prop. 4.2].

Below, we record three properties of harmonic functions in Lipschitz domains. The first one is very well known:

cvntp1 **Lemma 4.3.** *Let u be harmonic in $\Omega^\pm$ and such that $\mathcal{N}_\alpha^{\Omega^\pm} u(\xi) < \infty$ for $\xi \in E \subset \mathcal{M}$. Then, u has nontangential limit from $\Omega^\pm$ at σ-a.e. $\xi \in E$.*

For the bounded domain Ω^+, Lemma 4.3 follows from the combination of [34, Sec. 5, Thm] and [11, Thm. 1]. The case of Ω^- reduces to the one of a bounded domain by excising out the complement of a large ball. Let us mention that the lemma is valid more generally on the class of nontangentially accessible domains, see [14, Thm. 6.4].

The second property that we set forth is a global version of results from [10].

Dirichlet1 **Theorem 4.4.** *For $2 \le p \le \infty$, to each $\varphi \in L^p(\mathcal{M})$ there is a unique harmonic function $u_\varphi^\pm : \Omega^\pm \to \mathbb{R}$ (including at infinity in the case of Ω^-) converging nontangentially to $\varphi(\xi)$ from $\Omega^\pm$ at σ-a.e. $\xi \in \mathcal{M}$, and such that $\|\mathcal{N}_\alpha^{\Omega^\pm} u_\varphi^\pm\|_{L^p(\mathcal{M})} < \infty$. It holds that $u_\varphi^\pm(z) = \int \varphi d\omega_z^\pm$, moreover there is a constant $C = C(\alpha, p, \Omega)$ such that $\|\mathcal{N}_\alpha^{\Omega^\pm} u_\varphi^\pm\|_{L^p(\mathcal{M})} < C\|\varphi\|_{L^p(\mathcal{M})}$. If $u^\pm$ is harmonic in $\Omega^\pm$ (including at infinity in the case of Ω^-) and $\|\mathcal{N}_\alpha^{\Omega^\pm} u^\pm\|_{L^p(\mathcal{M})} < \infty$, then $u^\pm = u_\varphi^\pm$ for some $\varphi \in L^p(\mathcal{M})$.*

Proof. Let $\varphi \in L^p(\mathcal{M})$ and put $u_\varphi^\pm(x) := \int \varphi d\omega_x^\pm$. As $p \geq 2$, we get from Lemma 4.2 that $u_\varphi^\pm$ is well-defined and harmonic in $\Omega^\pm$. Let $\{C_{\theta,\pm z}\}$ be a regular family of cones, associated to a G-atlas $(U_j,\phi_j)_{1\leq j\leq N}$ of $\mathcal{M}$, with cover V_j such that $\overline{V_j} \subset U_j$, see Lemma 7.8. For $\xi \in V_j$ and θ_2 as in (49), consider the natural cones $C_{\theta_2,v_j^+}(\xi)$ and put $M_{\theta_2}u_\varphi^+(\xi) := \sup_{x\in C_{\theta_2,v_j^+}(\xi)} |u_\varphi^+(x)|$. When $2 \leq p < \infty$, it follows from [10, Thm.2] that $\int_{V_j} M_{\theta_2}u_\varphi^+ d\sigma \leq C_1\|\varphi\|_{L^p(\mathcal{M})}$ for some $C_1 = C_1(\mathcal{M}, V_j, \theta_2, p)$, and summing over j we get, since the V_j cover $\mathcal{M}$, that $\|N_{\theta,z}^\pm u_\varphi^+\|_{L^p(\mathcal{M})} \leq C_2\|\varphi\|_{L^p(\mathcal{M})}$, whence $\|\mathcal{N}_\alpha^{\Omega^+}u_\varphi^+\|_{L^p(\mathcal{M})} \leq C\|\varphi\|_{L^p(\mathcal{M})}$ where $C = C(\alpha, p, \Omega)$; if $p = \infty$, the corresponding estimate holds with $C = 1$ as ω_x^+ is a probability measure, absolutely continuous with respect to σ. As $(u_\varphi^+)_{|\mathcal{M}} = \varphi$ when φ is continuous, and since continuous functions are dense in $L^2(\mathcal{M})$ where any converging sequence has a subsequence converging pointwise a.e., the previous estimate implies that u_φ^+ has nontangential limit φ from Ω^+ at σ-a.e. point of $\mathcal{M}$, as soon as $\varphi \in L^2(\mathcal{M})$. Conversely, let u be harmonic in Ω^+ with $\|\mathcal{N}_\alpha^{\Omega^+}u\|_{L^p(\mathcal{M})} < \infty$ for some $p \in [2,\infty)$. Pick $j_1 \in \{1,\cdots,N\}$ and some open set $W_{j_1} \subset U_{j_1}$ in $\mathcal{M}$ with $\overline{V}_{j_1} \subset W_{j_1} \subset \overline{W}_{j_1} \subset U_{j_1}$. By the uniform cone property discussed after (19), there is $\beta > 1$ such that $\xi + \varepsilon v_{j_1}^+ \in R_\beta^{\Omega^+}(\xi)$ for $\varepsilon > 0$ small enough and $\xi \in W_{j_1}$. By Lemma 4.1 $\|\mathcal{N}_\beta^{\Omega^+}u\|_{L^p(\mathcal{M})} \leq c < \infty$, so $\int_{W_{j_1}} |u(\xi + \varepsilon v_{j_1}^+)|^p d\sigma(\xi) < c$ *a fortiori*. Then, [10, Thm. 3] implies there is $f_{j_1} \in L^p(\mathcal{M})$ such that $\lim_{\Omega^+\ni x\to\xi} u_{f_{j_1}}^+(x) - u(x) = 0$ for *every* $\xi \in \overline{V_{j_1}}$ (the decisive fact here is that the limit needs *not* be nontangential). Define a harmonic function h_{j_1} in Ω^+ by $h_{j_1}(x) := \int f_{j_1}\chi_{\mathcal{M}\setminus V_{j_1}} d\omega_x^+$. By Lemma 7.11, it holds that $\lim_{\Omega^+\ni x\to\xi} h_{j_1}(x) = 0$ for every $\xi \in V_{j_1}$. Thus, if we put $\psi_1 := f_{j_1}\chi_{V_{j_1}} \in L^p(\mathcal{M})$, we have that $\lim_{\Omega^+\ni x\to\xi} u_{\psi_1}^+(x) - u(x) = 0$ for $\xi \in V_{j_1}$, and by Lemma 7.11 again $\lim_{\Omega^+\ni x\to\xi} u_{\psi_1}^+(x) = 0$ for $\xi \in \mathcal{M} \setminus \overline{V}_{j_1}$.
Next, choose $j_2 \neq j_1$ such that $V_{j_2} \cap V_{j_1} \neq \emptyset$ and repeat the previous argument with V_{j_1} replaced by $V_{j_2} \setminus \overline{V_{j_1}}$ (note that connectedness was not an issue) to obtain a function $\psi_2 \in L^p(\mathcal{M})$, vanishing outside $V_{j_2} \setminus \overline{V_{j_1}}$, such that $\lim_{\Omega^+\ni x\to\xi} u_{\psi_2}^+(x) - u(x) = 0$ for $\xi \in V_{j_2} \setminus \overline{V_{j_1}}$ and $\lim_{\Omega^+\ni x\to\xi} u_{\psi_2}^+(x) = 0$ when $\xi \in \mathcal{M} \setminus \overline{V_{j_2} \setminus \overline{V_{j_1}}}$. Continuing in this fashion, we get a sequence of at most N disjoint open sets $O_1 := V_{j_1}$, $O_2 := V_{j_2} \setminus \overline{V_{j_1}}$, $O_3 := V_{j_3} \setminus \overline{V_{j_2} \cup V_{j_1}}$ ... that cover $\mathcal{M}$ up to a set $E \subset \cup_i \partial O_i$ of σ-measure zero, along with functions $\psi_i \in L^p(\mathcal{M})$ such that ψ_i vanishes outside O_i and $\lim_{\Omega^+\ni x\to\xi} u_{\psi_i}^+(x) - u(x) = 0$ for $\xi \in O_i$ while $\lim_{\Omega^+\ni x\to\xi} u_{\psi_i}^+(x) = 0$ when $\xi \in \mathcal{M} \setminus \overline{O_i}$. Set $\psi := \sum_i \psi_i$ and observe that $\lim_{\Omega^+\ni x\to\xi} u_\psi^+(x) - u(x) = 0$ for $\xi \in \cup_i O_i$, whence ψ is

equal σ-a.e. to the nontangential limit of u, by Lemma 4.3 and the first part of the proof. In view of Lemma 4.2, this shows in particular that u_ψ^+ is independent of the atlas, and we may pick the latter so that a given given $\xi \in \mathcal{M}$ belongs to exactly one set V_j and to the closure of no other, see Remark 1. As E is contained in $\cup_j \partial V_j$ by construction, this entails that $\lim_{\Omega^+ \ni x \to \xi} u_\psi^+(x) - u(x) = 0$ for *all* $\xi \in \mathcal{M}$, whence $u = u_\psi^+$ by the maximum principle. This achieves the proof for the bounded domain Ω^+, and the case of Ω^- follows from this one by inversion and Kelvin transform, see the discussion before (50).

□

The third result that we need is a classical estimate for layer potentials, often stated for bounded domains only. Recall the single layer potential of $h \in L^1(\mathcal{M})$, given by

$$\mathcal{S}h(x) = \frac{1}{\gamma_n(1-n)} \int_{\mathcal{M}} \frac{h(\zeta)}{|\zeta - x|^{n-1}} \, d\sigma(\zeta), \qquad x \in \mathbb{R}^{n+1} \setminus \mathcal{M}, \tag{51}$$ slp

where $\gamma_n = \mathcal{H}^n(\mathbb{S}^n)$. One can verify that the nontangential limit of $\mathcal{S}h$ from $\Omega^\pm$ at σ-a.e. $\xi \in \mathcal{M}$ is

$$Sh(\xi) = \frac{1}{\gamma_n(1-n)} \int_{\mathcal{M}} \frac{h(\zeta)}{|\zeta - \xi|^{n-1}} \, d\sigma(\zeta), \qquad \xi \in \mathcal{M}, \tag{52}$$ sslp

where we note that the integral is absolutely convergent.
For v a harmonic function on $\Omega^\pm$, the Lusin integral function $L_v : \mathcal{M} \to \mathbb{R}$ is defined by

$$L_v(\xi) := \left(\int_{C_{\theta, \pm z(\xi)}(\xi)} |\nabla v(x)|^2 |x - \xi|^{1-n} dm_{n+1}(x) \right)^{1/2}, \qquad \xi \in \mathcal{M}. \tag{53}$$ Lusindef

regL **Theorem 4.5.** *There exists $p_1 \in (2, \infty]$ such that, if $g \in W^{1,p}(\mathcal{M})$ for some $p \in (1, p_1)$, then there is a unique harmonic function $v_g^\pm$ in $\Omega^\pm$ with $\|\mathcal{N}_\alpha^{\Omega^\pm} \nabla v_g^\pm\|_{L^p(\mathcal{M})} < \infty$ and such that $v_g^\pm$ converges nontangentially to $g(\xi)$ from $\Omega^\pm$ at σ-a.e. $\xi \in \mathcal{M}$. Moreover, $S : L^p(\mathcal{M}) \to W^{1,p}(\mathcal{M})$ is invertible and*

$$v_g^\pm(x) = \mathcal{S}(S^{-1}g)(x), \qquad x \in \Omega^\pm. \tag{54}$$ repslp1

In addition, there is a constant $C = C(p, \Omega, \alpha)$ such that

$$\|\mathcal{N}_\alpha^{\Omega^\pm} v_g^\pm\|_{L^p(\mathcal{M})} + \|\mathcal{N}_\alpha^{\Omega^\pm} \nabla v_g^\pm\|_{L^p(\mathcal{M})} < C \|g\|_{W^{1,p}(\mathcal{M})}. \tag{55}$$ inegSobnt

If $\mathcal{M}$ is C^1-smooth, then we can take $p_1 = \infty$.

Proof. On Ω^+, the result follows from [54, Thm. 5.1 & Rem. 5.9] or [13, Thms. 4.14, 4.17 & 4.18], anf when $\mathcal{M}$ is C^1-smooth from [19, Thm. 2.4], except for the inequality $\|\mathcal{N}_\alpha^{\Omega^+} v_g^+\|_{L^p(\mathcal{M})} < C\|g\|_{W^{1,p}(\mathcal{M})}$. To obtain the latter, it is enough to assume that $v_g^+(x_0) = 0$ for some fixed $x_0 \in \Omega$, because $|v_g(x_0)| \leq c\|g\|_{L^p(\mathcal{M})}$ for some $c = c(x_0, p, \Omega)$, by (54). Then, we get from [12, Thm. 1] that $\|L_{v_g^+}\|_{L^p(\mathcal{M})}$ and $\|\mathcal{N}_\alpha^{\Omega^+} v_g\|_{L^p(\mathcal{M})}$ are equivalent quantities, with constants depending on Ω, p and x_0. By inspection, $|L_{v_g^+}| \leq C'\mathcal{N}_\alpha^{\Omega^+} \nabla v_g^+$ pointwise for suitable α and C' (remember (49)), which gives us what we want.

To obtain the result on the unbounded domain Ω^-, we assume that $0 \in \Omega^+$ and appeal again to the inversion $\mathcal{I}$ and the Kelvin transform K, as in the proof of Theorem 4.4. Specifically, we apply the result just proven to $\Omega_1 := \mathcal{I}(\Omega^-)$ and $\widetilde{g} := K[g]$, observing that $\|\widetilde{g}\|_{W^{1,p}(\partial\Omega_1)}$ and $\|g\|_{W^{1,p}(\mathcal{M})}$ are equivalent quantities. This yields a harmonic function $v_{\widetilde{g}}$ on Ω_1, and then $v_g^- := K[v_{\widetilde{g}}]$ satisfies all our requirements because

$$\nabla K[v_{\widetilde{g}}](x) = \frac{\nabla v_{\widetilde{g}}(\mathcal{I}(x))}{|x|^{n+1}} - 2\,\langle x\,,\, \nabla v_{\widetilde{g}}(\mathcal{I}(x))\rangle\, \frac{x}{|x|^{n+3}} - (n-1)v_{\widetilde{g}}(\mathcal{I}(x))\, \frac{x}{|x|^{n+1}}\,.$$

The uniqueness of v_g^- follows from the uniqueness of $v_{\widetilde{g}}$.

□

5. Hardy spaces of harmonic gradients

hargrad

For $1 \leq p \leq \infty$, we define the Hardy space $\mathcal{H}^p_\pm$ to consist of vector fields $F : \Omega^\pm \to \mathbb{R}^{n+1}$ such that $\mathcal{N}_\alpha^{\Omega^\pm} F \in L^p(\mathcal{M})$ and $F = \nabla u$ for some real-valued function u which is harmonic in $\Omega^\pm$ (including at infinity in the case of Ω^-). The value of α is immaterial, by Lemma 4.1. Endowed with the L^p norm of $\mathcal{N}_\alpha^{\Omega^\pm} F$, one sees that $\mathcal{H}^p_\pm$ is a Banach space.

Hardy spaces of harmonic gradients were first introduced on half-spaces in [52], and studied over C^1-domains in [18, 48]. On domains whose boundary is connected and has the local Lipschitz graph property, they are implicitely considered in [54] for $1 < p < 2 + \varepsilon$ and more explicitly in [13] for $p = 1$, as companions to regularity theory of the Laplacian; here, $\varepsilon > 0$ depends on $\mathcal{M}$. Since harmonic gradients identify with vector-valued Clifford-analytic functions (see Lemma 7.6), $\mathcal{H}_p^\pm$ may also be viewed as a closed subspace of the Clifford-analytic Hardy space of exponent p studied when $1 < p < \infty$ over Lipschitz graphs in [41, Ch. 4] and, more generally, on non-tangentially

accessible domains with Ahlfors-David regular boundary in [33, sec. 4], see Section 7.3 for a definition of Clifford-analyticity.
Hereafter our main concern is the range $1 < p < \infty$, but we briefly discuss the cases $p = 1, \infty$ for completeness.

propHp **Proposition 1.** *Let $\mathcal{M} \subset \mathbb{R}^{n+1}$ be a compact and connected hypersurface with the local Lipschitz graph property, and $p \in [1, \infty]$. Every $F \in \mathcal{H}^p_\pm$ has a nontangential limit $F^*(\xi)$ at σ-a.e. $\xi \in \mathcal{M}$. The function F^* characterizes F uniquely. Moreover, for $1 < p \leq \infty$ to each $\alpha > 1$, there is a constant $C = C(\alpha, p, \mathcal{M})$ such that*

$$\|\mathcal{N}^{\pm}_{\alpha} F\|_{L^p(\mathcal{M})} \leq C \|F^*\|_{L^p(\mathcal{M}, \mathbb{R}^{n+1})}. \tag{56}$$ nntl

Proof. The existence of F^* is clear from Lemma 4.3. Because $\mathcal{M}$ is compact and has the local Lipschitz graph property, one can see that $\Omega^\pm$ is a particular instance of a so-called nontangentially accessible domain, see [14] and [33] for the definitions. Therefore, observing from Lemma 7.6 that $\mathcal{H}^p_\pm$ is a subspace of the Clifford -analytic Hardy space defined in [33, Def. 4.7.3], the case $1 < p < \infty$ follows from [33, Eqns. 4.7.13 & 4.7.11]. The case $p = \infty$ follows from Lemma 4.4, applied componentwise. When $p = 1$ and $F \in \mathcal{H}^1_+$, it follows from [13, Thm. 4.12] that $F = \nabla u$ where the nontangential limit u^* (it lies in $L^1(\mathcal{M})$ by [12, Thm. 1] and the definition of L_u in (53)), when normalized to have mean zero, lies in the so denoted space $H^1_{1,at}(\mathcal{M})$ with

$$\|u^*\|_{H^1_{1,at}(\mathcal{M})} \leq C_1 \|\mathcal{N}^{\Omega^+}_{\alpha} F\|_{L^1(\mathcal{M})} \leq C_2 \|u^*\|_{H^1_{1,at}(\mathcal{M})},$$

and u (thus, also F) is uniquely defined by its nontangential limit. The case where $F \in \mathcal{H}^1_-$ follows by Kelvin transform, as in the proof of Theorem 4.5. □

In view of Proposition 1, we may as well equip $\mathcal{H}^\pm_p$ with the $L^p(\mathcal{M})$-norm of its nontangential limit when $1 < p \leq \infty$, thereby identifying this Hardy space with a closed subset of $L^p(\mathcal{M}, \mathbb{R}^{n+1})$.
Let us point out that [33, Eqns. 4.7.13 & 4.7.11], while a convenient reference to us, conceals that Proposition 1 does not depend on the fact that F is a gradient when $p \geq 2$, because then everything follows from Theorem 4.4. It is when $1 \leq p < 2$ that this fact becomes decisive. Also, it would be interesting to know if (56) holds when $p = 1$.
The case $1 < p \leq 2$ of Proposition 1 would alternatively follow without much difficulty from Theorem 4.5 and the next lemma that will of use later on.

derGT **Lemma 5.1.** *For $1 < p < \infty$, each $F \in \mathcal{H}^p_\pm$ is of the form $F = \nabla f$ where f is harmonic in $\Omega^\pm$ (including at infinity in the case of Ω^-) with nontangential limit $f^* \in W^{1,p}(\mathcal{M})$ such that ∇f^* is the tangential component of F^* at σ-a.e. point of $\mathcal{M}$.*

Proof. We only consider the case of Ω^+, as the case of Ω^- will then follow by Inversion and Kelvin transform. Fix $\zeta_0 \in \Omega^+$ and let f be a harmonic function in Ω^+ such that $\nabla f = F$, with $f(\zeta_0) = 0$. Argueing with the Lusin integral function as we did in the proof of Theorem 4.5, we find that $\mathcal{N}_\alpha^{\Omega^+} f \in L^p(\mathcal{M})$. Hence, f has nontangential limit f^* at σ-a.e. point of $\mathcal{M}$. Let $\{C_{\theta,\pm z}\}$ be a regular family of cones, associated to a G-atlas $(U_j, \phi_j)_{1\leq j\leq N}$ of $\mathcal{M}$ whose coordinate cylinders $\mathcal{C}_j$ have cross-section a ball, and such that the $V_j := \mu\mathcal{C}_j \cap \mathcal{M}$ are a cover for the family $\{C_{\theta,\pm z}\}$ for some $0 < \mu < 1$, see Lemma 7.8. Recall also the notation $B_j = \phi_j(U_j)$, so that $\phi_j(V_j) = \mu B_j$. Fix $j \in \{1, \cdots, N\}$, and assume for simplicity that $\mathcal{C}_j$ is oriented along the x_{n+1}-axis so as to write $\xi = (y, \Psi_j(y))$ when $\xi \in V_j$, for some unique $y \in \mu B_j$. For $\varepsilon > 0$ small enough that $(y, \Psi_j(y) - \varepsilon) \in \Omega^+$ when $y \in \mu B_j$, the smoothness of f in Ω^+ implies that $h_\varepsilon(y) := f(y, \Psi_j(y) - \varepsilon)$ is Lipschitz in μB_j with $(\nabla h_\varepsilon(y))^t = (F(y, \Psi_j(y) - \varepsilon))^t D\phi_j^{-1}(y)$. Let $\varepsilon_k \to 0$ and observe from (49) that $(y, \Psi_j(y) - \varepsilon_k)$ converges to $\xi = (y, \Psi_j(y))$ from within $C_{\theta,-z(\xi)}(\xi)$, hence $F(y, \Psi_j(y) - \varepsilon_k)$ converges for m_n-a.e. $y \in \mu B_j$ to $F^*(y, \Psi_j(y))$. Moreover, it is less than $N^+_{\theta,z} F \circ \phi_j^{-1}(y)$ which lies in $L^p(\mu B_j)$, thanks to (10), (11) and our assumption that $F \in \mathcal{H}^p_+$. Therefore, by dominated convergence, ∇h_{ε_k} converges in $L^p(\mu B_j)$ to $(D\phi_j^{-1})^t F^* \circ \phi_j^{-1}$. Besides, h_{ε_k} converges pointwise a.e. to $f^*(y, \Psi_j(y))$, and clearly $\|h_{\varepsilon_k}\|_{L^p(\mu B_j)} \leq \|N^+_{\theta,z} f \circ \phi_j^{-1}\|_{L^p(\mu B_j)}$ which is finite. Hence, replacing ε_k by a subsequence if necessary, we may assume that h_{ε_k} converges weakly in $L^p(\mu B_j)$, and since the pointwise and the weak limit coincide there when both exist we find that $f^* \circ {\phi_j^{-1}}_{|\mu B_j}$ lies in $W^{1,p}(\mu B_j)$ and has Euclidean gradient:

$$\nabla(f^* \circ {\phi_j^{-1}}_{|\mu B_j}) = (D\phi_j^{-1})^t F^* \circ (\phi_j^{-1})_{|\mu B_j}. \tag{57}$$ expglim

As the V_j cover $\mathcal{M}$, this exactly means that ∇f^* is the orthogonal projection of F^* onto $T_\xi\mathcal{M}$ for σ-a.e. ξ. □

6. The Hardy Hodge decomposition

HaHodecs

As in the previous section, let $\mathcal{M}$ be a compact connected hypersurface embedded in $\mathbb{R}^{n+1}$ with the local Lipschitz graph property. Below, we prove a

decomposition for vector fields $\mathcal{M} :\to \mathbb{R}^{n+1}$ that combines the Helmholtz-Hodge decomposition and Hardy spaces of harmonic gradients. This decomposition that we shall call the Hardy-Hodge decomposition, generalizes the familiar decomposition of a complex function on a plane curve as the sum of a holomorphic function in the Hardy class of the interior of the curve and of a holomorphic function in the Hardy class the exterior of the curve. In higher dimension, a third term is needed because the tangential component of a harmonic gradient is a gradient vector field on $\mathcal{M}$, whereas not evry tangent vector field is a gradient in dimension greater than 1.
The Hardy-Hodge decomposition has interesting applications to inverse potential problems with source term in divergence form, like inverse magnetization problems. On the plane and the sphere (where the decomposition is already known to exist), some of them can be found in [7, 22]; on more general surfaces, *cf.* the manuscript [23].

HHdec **Theorem 6.1** (Hardy-Hodge decomposition)**.** *Let $\mathcal{M} \subset \mathbb{R}^{n+1}$ be a compact and connected hypersurface with the local Lipschitz graph property. Let $\mathcal{D}^p(\mathcal{M})$ designate the tangential divergence free vector fields of L^p-class on $\mathcal{M}$. Then, there exist $\varepsilon, \varepsilon' > 0$ such that, for $2-\varepsilon < p < 2+\varepsilon'$, it holds the direct sum:*

$$L^p(\mathcal{M}, \mathbb{R}^n) = \mathcal{H}_p^+ \oplus \mathcal{H}_p^- \oplus \mathcal{D}^p(\mathcal{M}). \tag{58}$$ HHdece

Moreover, if $\mathcal{M}$ is C^1-smooth then (58) *holds for all $p \in (1, \infty)$.*

Proof. Let $V \in L^p(\mathcal{M}, \mathbb{R}^n)$. Let us write $V = V_n + V_t$ where V_n is the normal component and V_t the tangential component of V. By Theorem 3.1, it holds for some $\varepsilon \in (0, 1/2)$ and $2/(1+2\varepsilon) < p < 2/(1-2\varepsilon)$ that $V_t = G + D$, where $D \in \mathcal{D}^p(\mathcal{M})$ and $G \in \mathcal{G}^p(\mathcal{M})$ is the gradient of some function $\psi \in W^{1,p}(\mathcal{M})$. Let u be the harmonic function in Ω^+ solving the Dirichlet problem with boundary condition $u_{|\mathcal{M}} = \psi$; decreasing the upper bound $2/(1-2\varepsilon)$ on p to $p_1 > 2$ if needed, u indeed exists with $\nabla u \in \mathcal{H}_+^p$, by Theorem 4.5. Then, Lemma 5.1shows that $V - D - \nabla u$ has zero tangential component on $\mathcal{M}$, hence we are left to decompose *normal* vector fields in $L^p(\mathcal{M}, \mathbb{R}^n)$. Now, if hn is such a field, where $h \in L^p(\mathcal{M})$ and n is the unit outer normal field on $\mathcal{M}$, then the single layer potential $\mathcal{S}h(x)$ given by (51) defines a harmonic function $u^\pm$ in $\partial\Omega^\pm$ whose nontangential limit Sh on $\mathcal{M}$ from either side is given by (52) and lies in $W^{1,p}(\mathcal{M}$, by Theorem 4.5 again; moreover, the same theorem tells us that $\nabla u^\pm$ lies in $\mathcal{H}^\pm$, and classical jump conditions imply that the nontangential limits from each side satisfy $(\nabla u^-)^* - (\nabla u^+)^* = hn$; see, *e.g* [54, Thm. 1.11]. This shows that $L^p(\mathcal{M}, \mathbb{R}^n) = \mathcal{H}_p^+ + \mathcal{H}_p^- + \mathcal{D}^p(\mathcal{M})$.

To see that the sum is direct, assume that $F^+ + F^- + D = 0$ for some $F^+ \in \mathcal{H}^p_+$, $F^- \in \mathcal{H}^p_-$, and $D \in \mathcal{D}^p(\mathcal{M})$. Because the tangential components of $(F^+)^*$ and $(F^-)^*$ are gradients by Lemma 5.1, we necessarily have that $D = 0$ because no nonzero member of $\mathcal{D}^p(\mathcal{M})$ can be a gradient. Thus, $(F^+)^* + (F^-)^* = 0$. Let $u^\pm$ be harmonic and real-valued in $\Omega^\pm$ with $\nabla u^\pm = F^\pm$. By Lemma 5.1 the gradients of $(u^+)^*$ and $(-u^-)^*$ agree on $\mathcal{M}$, hence we may assume that $(u^+)^* = (-u^-)^*$. Now, since the normal components of $(\nabla u^+)^*$ and $(-\nabla u^-)^*$ also agree, the distribution which coincides with u^+ on Ω^+ and $-u^-$ on Ω^- is harmonic on $\mathbb{R}^{n+1}$, by the Green formula, hence it is a harmonic function. Because it vanishes at infinity, this function is identically zero by Liouville's theorem. Finally, when $\mathcal{M}$ is C^1-smooth, it is *a fortiori* VMO-smooth and both Theorem 3.1 and Theorem 4.5 apply. □

Note that the set of p for which (58) holds contains the interval $(2/(1+2\varepsilon, p_1)$, where ε is as in Theorem 3.1 and p_1 as in Theorem 4.5.

7. Appendix

app

VMOa *7.1. BMO, VMO and BMO_L functions on open sets.*

For $\Omega \subset \mathbb{R}^k$ an open set, the spaces $BMO(\Omega)$, $VMO(\Omega)$ and $VMO_{loc}(\Omega)$ were defined Section 1.3. For $E \subset \mathbb{R}^k$ and $g \in L^1(E)$, recall also the elementary inequality valid for each $b \in \mathbb{R}$:

$$\int_E |g - g_E|\, dm_k \leq 2 \int_E |g - b|\, dm_k. \tag{59}$$ elm

invcompL **Lemma 7.1.** *Let $\varphi : \Omega_1 \to \Omega_2$ be a bi-Lipchitz map between open subsets of $\mathbb{R}^k$. Then, $h \mapsto h \circ \varphi$ is an isomorphism from $VMO(\Omega_2, \mathbb{R}^k)$ onto $VMO(\Omega_1, \mathbb{R}^k)$.*

Proof. For $B(x_1, \rho_1) \subset \Omega_1$ and $B(x_2, \rho_2) \subset \Omega_2$, it holds that

$$B(\varphi(x_1), \rho_1/\mathfrak{c}_{\varphi^{-1}}) \subset \varphi(B(x_1, \rho_1)) \subset B(\varphi(x_1), \rho_1 \mathfrak{c}_\varphi), \tag{60}$$ dilat

Assume without loss of generality that $\mathfrak{c}_\varphi \geq 1$. Let $\tau = \mathfrak{c}_\varphi \mathfrak{c}_{\varphi^{-1}} \geq 1$, and $B(a, r)$ be such that $B(a, \tau r) \subset \Omega_1$. We see from the first inclusion in (60) that $B(\varphi(a), r\mathfrak{c}_\varphi) \subset \varphi(B(a, \tau r)) \subset \Omega_2$, and since $|J_k \varphi| \geq \mathfrak{c}^{-k}_{\varphi^{-1}}$, by (3), the change of variable formula and the second inclusion in (60) yield:

$$\begin{aligned} \int_{B(a,r)} |h \circ \varphi - h_{B(\varphi(a), r\mathfrak{c}_\varphi)}| dm_k &\leq \mathfrak{c}^k_{\varphi^{-1}} \int_{\varphi(B(a,r))} |h - h_{B(\varphi(a), r\mathfrak{c}_\varphi)}| dm_k \\ &\leq \mathfrak{c}^k_{\varphi^{-1}} \int_{B(\varphi(a), r\mathfrak{c}_\varphi)} |h - h_{B(\varphi(a), r\mathfrak{c}_\varphi)}| dm_k. \end{aligned} \tag{61}$$ isoBMOL

Now, (59) and (61) imply that

$$\frac{1}{m_k(B(a,r))}\int_{B(a,r)}|h\circ\varphi - h_{B(a,r)}|dm_k \leq 2\tau^k\|h\|_{BMO(B(\varphi(a),r\mathfrak{c}_\varphi))},$$

and since $r\mathfrak{c}_\varphi$ goes to zero with r the conclusion follows. $\square$

multinvVMO **Lemma 7.2.** *For $\Omega \subset \mathbb{R}^k$ an open set, $L^\infty(\Omega)\cap VMO(\Omega)$ is an algebra. If $h \in L^\infty(\Omega)\cap VMO(\Omega)$ and $h \geq \eta > 0$, then $h^{1/2}$ and $1/h$ belong to $L^\infty(\Omega)\cap VMO(\Omega)$.*

Proof. This follows immediately from the inequalities:

$$\begin{aligned}
&\int_Q |hg - h_Q g_Q|\, dm_k \leq \|h\|_{L^\infty(\Omega)}\int_Q |g-g_Q|\, dm_k + \|g\|_{L^\infty(\Omega)}\int_Q |h-h_Q|\, dm_k,\\
&\int_Q |h^{1/2} - h_Q^{1/2}|\, dm_k = \int_Q \tfrac{|h-h_Q|}{h^{1/2}+h_Q^{1/2}}\, dm_k \leq \tfrac{1}{2\eta^{1/2}}\int_Q |h-h_Q|\, dm_k,\\
&\int_Q |h^{-1} - h_Q^{-1}|\, dm_k = \int_Q \tfrac{|h-h_Q|}{hh_Q}\, dm_k \leq \tfrac{1}{\eta^2}\int_Q |h-h_Q|\, dm_k,
\end{aligned}$$

$\square$

compVMO1i **Lemma 7.3.** *Let $\varphi : \Omega_1 \to \Omega_2$ be a bi-Lipschitz map between open subsets of $\mathbb{R}^k$, and assume in addition that $\varphi \in VMO^{1,\infty}(\Omega_1,\mathbb{R}^k)$. If $h \in VMO^{1,\infty}(\Omega_2)$, then $h\circ\varphi \in VMO^{1,\infty}(\Omega_1)$.*

Proof. This follows from Lemma 7.1, Lemma 7.2 and the chain rule. $\square$

7.2. Compact VMO-smooth manifolds

Recall from Section 1.5 that a compact Lipschitz manifold is VMO-smooth if it has a B-atlas $\cup_j(U_j,\phi_j)_{1\leq j\leq N}$, in the sense of Section 1.4, whose changes of charts $\phi_{j_1}\circ\phi_{j_2}^{-1}$ lie in $VMO^{1,\infty}(\phi_{j_2}(U_{j_1}\cap U_{j_2}),\mathbb{R}^n)$ for all j_1, j_2. Such an atlas is called a $VMO^{1,\infty}$-atlas.

Below, we adapt the proof of [43, Thm. 4.2] to obtain an embedding result for compact VMO-smooth manifolds:

paramV **Lemma 7.4.** *Let $\mathcal{M}$ be a VMO-smooth manifold of dimension $n \geq 2$. There is a bi-Lipschitz embedding $f : \mathcal{M} \to \mathbb{R}^m$ with $m \leq (n+1)^2$, and a VMO-smooth atlas (V_l,ψ_l) on $f(\mathcal{M})$ such that each ψ_l is bi-Lipschitz and $\psi_l^{-1} : \psi_l(V_l) \to \mathbb{R}^m$ lies in $VMO^{1,\infty}_{loc}(\psi_l(V_l),\mathbb{R}^m)$.*

Proof. Let (W_ℓ, η_ℓ) be a VMO-smooth atlas. Refining the latter if necessary, we may assume since $\mathcal{M}$ is a locally compact metric space that each W_ℓ has compact closure, that $\eta_\ell(W_\ell)$ is bounded and that the cover (W_ℓ) is locally finite. Refining further is needed, we can write $\mathcal{M} = \mathcal{B}_0 \cup \cdots \cup \mathcal{B}_s$, where $\mathcal{B}_k = \cup_{i\in I_k} U_{i,k}$ is a union of disjoint open sets $U_{i,k}$, each of which is a W_ℓ, with $s \le n$ and I_k a subset of the positive integers [42, Lem. 2.7]. If, say $U_{i,k} = W_\ell$, we put $\phi_{i,k} = \eta_\ell$. Composing η_ℓ with a non-singular linear map, we can also ensure that $\phi_{i,k}(U_{i,k})$ is contained in the ball $B(3ie_1, 1)$ with $e_1 = (1, 0, \cdots, 0)^t$, and this does not alter the VMO-smoothness of the atlas by Lemmas 7.1 and 7.3. Pick $(O_{i,k})$ an open cover of $\mathcal{M}$ with $\overline{O}_{i,k} \subset U_{i,k}$. This is possible since $\mathcal{M}$ is normal. Let $h_{i,k} : \phi_{i,k}(U_{i,k}) \to [0,1]$ be a C^∞-smooth compactly supported function, identically 1 on the compact set $\overline{\phi_{i,k}(O_{i,k})}$. Define $\varphi_{i,k} : \mathcal{M} \to [0,1]$ to be $h_{i,k} \circ \phi_{i,k}$ on $U_{i,k}$ and 0 elsewhere, then set $\omega_k = \sum_{i\in I_k} \varphi_{k,i}$. Note that ω_k is well defined for at most one summand is nonzero at each $x \in \mathcal{M}$, and clearly $\omega_k : \mathcal{M} \to \mathbb{R}$ is Lipschitz with $\operatorname{supp} \omega_k \subset \mathcal{B}_k$. *We claim* that $\omega_k \circ \eta_\ell^{-1}$ belongs to $VMO^{1,\infty}_{loc}(\eta_\ell(W_\ell))$ for each ℓ. Indeed, it is equal to $h_{i,k} \circ (\phi_{i,k} \circ \eta_\ell^{-1})$ on the open set $B_{\ell,i,k} := \eta_\ell(W_\ell \cap U_{i,k})$, hence the restriction $(\omega_k \circ \phi_\ell^{-1})_{|B_{\ell,i,k}}$ lies in $VMO^{1,\infty}_{loc}(B_{\ell,i,k})$ because it is obviously bounded and locally Lipschitz while its derivative lies in $VMO_{loc}(B_{\ell,i,k})$, by the chain rule, Lemma 7.2 and the fact that (V_ℓ, ψ_ℓ) is a VMO-smooth atlas. So, when $y \in \psi_\ell(W_\ell)$ belongs to some $B_{\ell,i,k}$, it has a neighborhood $\mathcal{V}(y)$ such that $\omega_k \circ \eta_\ell^{-1}{}_{|\mathcal{V}(y)} \in VMO^{1\infty}(\mathcal{V}(y))$. Next, assume that y belongs to no $B_{\ell,i,k}$. Let $r_0 > 0$ be such that $\overline{B}(y, r_0) \subset \eta_\ell(W_\ell)$ and observe, since the cover W_ℓ of $\mathcal{M}$ is locally finite, that for r_0 small enough $B(y, r_0)$ meets at most finitely many $B_{\ell,i,k}$, say $B_{\ell,i_1,k_1}, \cdots, B_{\ell,i_N,k_N}$. If (y_j) is a sequence converging to y in $B(y, r_0)$, then the sequence $\phi_{i_l,k_l} \circ \eta_\ell^{-1}(y_j)$ has at most finitely many terms in the compact set $\operatorname{supp} h_{i_l,k_l} \subset \phi_{i_l,k_l}(W_\ell \cap U_{i_l,k_l})$, otherwise a subsequence would converge to some z_1 in $\phi_{i_l,k_l}(W_\ell \cap U_{i_l,k_l})$ and, since $\eta_\ell \circ \phi_{i_l,k_l}^{-1}$ is an isomorphism from $\phi_{i_l,k_l}(W_\ell \cap U_{i_l,k_l})$ onto $B_{k,i_l.k_l}$, the corresponding subsequence of (y_j) would converge to $y = \eta_\ell \circ \phi_{i_l,k_l}^{-1}(z_1) \in B_{\ell,i_l,k_l}$, a contradiction. Thus, there is $r > 0$ such that $\omega_k \circ \eta_\ell^{-1}(z) = 0$ for all $z \in B(y, r)$, hence $B(y, r)$ is a neighborhood of y such that, trivially, $\omega_k \circ \eta_\ell^{-1}{}_{|B(y,r)}$ belongs to $VMO^{1\infty}(B(y, r))$. *This proves the claim.*

Let $\widetilde{\phi}_k : \mathcal{B}_k \to \mathbb{R}^n$ coincide with $\phi_{i,k}$ on $U_{i,k}$, set $m = (n+1)(s+1)$ and define $f : \mathcal{M} \to \mathbb{R}^m$ by $f = (\omega_0, \omega_0\widetilde{\phi}_0, \omega_1, \omega_1\widetilde{\phi}_1, \cdots, \omega_s, \omega_s\widetilde{\phi}_s)$, where the product $\omega_k\widetilde{\phi}_k$ is extended by zero outside of $\mathcal{B}_k$. As in the proof of [43, Thm. 4.2], one checks that f is a locally bi-Lipschitz homeomorphism onto a closed subset of $\mathbb{R}^m$. If we put $V_l = f(W_l)$ and $\psi_l = \eta_l \circ f^{-1}$, we get an atlas for $f(\mathcal{M})$ such that $\psi_l(V_l \cap V_j) = \eta_l(W_l \cap W_j)$ and $\psi_j \circ \psi_l^{-1} = \eta_j \circ \eta_l^{-1}$, hence this atlas is VMO-smooth since (W_ℓ, η_ℓ) is a VMO-smooth atlas for $\mathcal{M}$. Moreover, on $\psi_l(V_l) = \eta_l(W_l)$, it holds that

$$\psi_l^{-1} = \big(\omega_0 \circ \eta_l^{-1}, (\omega_0 \circ \eta_l^{-1})(\widetilde{\phi}_0 \circ \eta_l^{-1}), \cdots, \omega_s \circ \eta_l^{-1}, (\omega_s \circ \eta_l^{-1})(\widetilde{\phi}_s \circ \eta_l^{-1})\big).$$

By a previous claim $\omega_k \circ \eta_l^{-1} \in VMO^{1,\infty}_{loc}(\eta_l(W_l))$ for each $k \in \{0, \cdots, s\}$, and since $\widetilde{\phi}_k \circ \eta_l^{-1}$ coincides with $\phi_{i,k} \circ \eta_l^{-1}$ on $B_{l,i,k}$, we have that $\widetilde{\phi}_k \circ \eta_l^{-1}{}_{|B_{l,i,k}}$ lies in $VMO^{1,\infty}_{loc}(B_{l,i,k}, \mathbb{R}^n)$. Since the latter is an algebra, we deduce that the restriction $(\omega_k \circ \eta_l^{-1})(\widetilde{\phi}_k \circ \eta_l^{-1})_{|B_{l,i,k}}$ in turn lies in $VMO^{1,\infty}_{loc}(B_{l,i,k}, \mathbb{R}^n)$, and the same arguments we used to prove the claim shows that $(\omega_k \circ \eta_l^{-1})(\widetilde{\phi}_k \circ \eta_l^{-1})$ belongs to $VMO^{1,\infty}_{loc}(\eta_l(W_l), \mathbb{R}^n)$. Therefore ψ_l^{-1} lies in $VMO^{1,\infty}_{loc}(\eta_l(W_l), \mathbb{R}^m)$, as desired. $\square$

VMOat **Lemma 7.5.** *Let $\mathcal{M}$ be a VMO-smooth manifold of dimension $n \geq 2$, endowed with a Riemannian metric Γ. If there is a VMO-smooth atlas $(V_l, \psi_l)_{l \in L}$ such that $(g^{(l)}_{i,k})$ lies in $VMO(\psi_l(V_l), \mathbb{R}^{n \times n})$ for each l, then this property holds for any compatible VMO-smooth atlas.*

Proof. Let (W_ℓ, η_ℓ) be another VMO-smooth atlas, and $(h^{(\ell)}_{i,k})$ the corresponding metric tensor. On $\eta_\ell(W_\ell \cap V_l)$, it holds that

$$(h^{(\ell)}_{i,k}) = \Big(D(\psi_l \circ \eta_\ell^{-1})\Big)^t \Big((g^{(l)}_{i,k}) \circ (\psi_l \circ \eta_\ell^{-1})\Big) D(\psi_l \circ \eta_\ell^{-1}). \qquad (62)$$ transmet

Since $\psi_l \circ \eta_\ell^{-1} \in VMO^{1,\infty}_{loc}(\eta_\ell(W_\ell \cap V_l))$ by assumption, $D(\psi_l \circ \eta_\ell^{-1})$ lies in $L^\infty_{loc}(\eta_\ell(W_\ell \cap V_l)) \cap VMO_{loc}(\eta_\ell(W_\ell \cap V_l))$. Moreover, as $\psi_l \circ \eta_\ell^{-1}$ is locally bi-Lipschitz, $(g^{(l)}_{i,k}) \circ \psi_l \circ \eta_\ell^{-1}$ lies in $VMO_{loc}(\eta_\ell(W_\ell \cap V_l))$, by Lemma 7.1, and is otherwise bounded, by (10). Now, $L^\infty_{loc}(\eta_\ell(W_\ell \cap V_l)) \cap VMO_{loc}(\eta_\ell(W_\ell \cap V_l))$ is an algebra in view of Lemma 7.2, so we conclude from (62) that $(h^{(\ell)}_{i,k})$ restricted to $\eta_\ell(W_\ell \cap V_l)$w belongs to $VMO_{loc}(\eta_\ell(W_\ell \cap V_l))$. Since the open sets $\eta_\ell(W_\ell \cap V_l)$ cover $\eta_\ell(W_\ell)$ as l ranges over L, the result follows. $\square$

Cliffords *7.3. Some Clifford analysis*

The Clifford algebra $\mathfrak{C}_m$ is the unital algebra generated over $\mathbb{R}$ by e_j, $1 \leq j \leq m$, subject to the relations $e_j^2 = -1$ and $e_i e_j = -e_j e_i$ when $i \neq j$. Clearly $\mathfrak{C}_m$ is a finite-dimensional vector space over $\mathbb{R}$, and a natural basis consists of all products $e_{j_1}\cdots e_{j_k}$ where $i_1 < \cdots < i_k$ and $0 \le k \le m$ (the empty product is 1 by convention). Thus, each $z \in \mathfrak{C}_m$ can uniquely be written in the form

$$z = x_0 + \sum_{k=1}^{m} \left(\sum_{1\le j_1<\cdots<j_k\le m} x_{j_1,\cdots,j_k}\; e_{j_1}e_{j_2}\cdots e_{j_k} \right), \tag{63}$$ decc

where x_0 and the $x_{j_1,\cdots,j_k}$ are real numbers. In (63), the homogeneous part of degree 0 with respect to the e_j, namely x_0, is called the scalar part of z, denoted by $\mathrm{Sc}\, z$, while the homogeneous part of degree 1, namely $x_1e_1+\cdots+x_me_m$, is called the vector part of z, denoted as $\mathrm{Vec}\, z$. The conjugate of z is

$$\overline{z} = x_0 + \sum_{k=1}^{m} (-1)^k \left(\sum_{1\le j_1<\cdots<j_k\le m} x_{j_1,\cdots,j_k}\; e_{j_1}e_{j_2}\cdots e_{j_k} \right), \tag{64}$$ deccc

and the norm of z is defined by

$$|z| = (z\overline{z})^{1/2} = \left(x_0^2 + \sum_{k=1}^{m} \sum_{1\le j_1<\cdots<j_k\le m} x_{j_1,\cdots,j_k}^2 \right)^{1/2}.$$

The subspace of $\mathfrak{C}_m$ comprised of homogeneous elements of degree 1 identifies isometrically with Euclidean space $\mathbb{R}^m$. Elements of this subspace are called vectors, so that z is a vector if and only if it reduces to its vector part.

The space of $\mathfrak{C}_m$-valued distributions on an open set $\mathcal{O} \subset \mathbb{R}^m$ is the tensor product $\mathfrak{C}_m \otimes \mathcal{D}(\mathcal{O})$, where $\mathcal{D}(\mathcal{O})$ indicates the distributions on $\mathcal{O}$. They act naturally on smooth $\mathfrak{C}_m$-valued functions with compact support in $\mathcal{O}$, but one must specify whether the action is taken from the left or the right because $\mathfrak{C}_m$ is a non-commutative algebra. The Dirac operator D is defined by

$$D = e_1\partial/\partial_{x_1} + e_2\partial/\partial x_2 + \cdots + e_n\partial/\partial x_m, \tag{65}$$ defD

see *e.g.* [33, Eq. 3.4.5]. It acts on $\mathfrak{C}_m$-valued distributions from the right and from the left by letting the differentiation operators commute with multiplication in $\mathfrak{C}_m$, but the two actions differ from each other. A distribution $f \in \mathfrak{C}_m \otimes \mathcal{D}(\mathcal{O})$ is called left Clifford-analytic if $Df = 0$ and right Clifford-analytic if $fD = 0$. Because $D^2 = -\Delta$ where $\Delta = \partial^2/\partial {x_1}^2 + \cdots + \partial^2/\partial {x_m}^2$ is the Euclidean Laplacian, every left or right Clifford-analytic distribution is a $\mathfrak{C}_m$-valued function with harmonic components in any basis of $\mathfrak{C}_m$, by Weyl's lemma.

The connection between harmonic gradients and Clifford-analytic functions stems from the following elementary observation.

connechg **Lemma 7.6.** *A vector-valued function defined on an open set $\mathcal{O} \subset \mathbb{R}^m$ is left Clifford-analytic if and only if it is right Clifford-analytic, and this is also if and only if it is locally the gradient of a harmonic function.*

Proof. This follows from a straightforward computation: assume f is left Clifford-analytic on $\mathcal{O}$ and set $f = \sum_{j=1}^m f_j$ where the f_j are real valued harmonic functions. By definition of left Clifford-analyticity, it holds that

$$0 = Df = -\sum_{j=1}^m \partial_{x_j} f_j + \sum_{1\le j_1<j_2\le m} (\partial_{x_{j_1}} f_{j_2} - \partial_{x_{j_2}} f_{j_1}) e_{j_1} e_{j_2}$$

which is equivalently to

$$\operatorname{div}(f_1, \cdots, f_m)^t = 0 \qquad \text{and} \qquad \operatorname{curl}\,(f_1, \cdots, f_m)^t = 0. \tag{66}$$ equival

The second equation in (66) implies that $(f_1, \cdots, f_m)^t$ is locally the gradient of a real-valued function Φ, and then the first equation says that $\Delta\Phi = 0$. Conversely, if $f_j = \partial_{x_j}\Phi$ for some locally defined harmonic function Φ, then (66) holds so that $Df = 0$, as desired. The case where f is right Clifford-analytic is similar. $\square$

If f is left Clifford-analytic in Ω^+ and $\mathcal{N}_\alpha^+|f|$ lies in $L^p(\mathcal{M})$ for some $p \in (1, \infty)$, then f has a nontangential limit a.e. on $\mathcal{M}$ from Ω^+, denoted by $f^+ \in L^p(\mathcal{M})$, and it holds (see [33, Eqns. 4.7.13 & 4.7.11]) that

$$f(z) = \frac{1}{4\pi} \int_{\mathcal{M}} \frac{\overline{y-z}}{|y-z|^3} n(y) f^+(y) d\sigma(y), \qquad z \in \Omega^+. \tag{67}$$ Cauchy+

In (67), $n(y)$ is the exterior unit normal to $\mathcal{M}$ at $y \in \mathcal{M}$ and the products are Cliffordian. If $z \in \Omega^-$, then the right hand side of (67) is equal to zero. Because (67) generalizes the Cauchy formula, we denote the integral in the right hand side of (67) by $\mathcal{C}f^+$.
A similar formula holds if f is left Clifford-analytic in Ω^- and $\mathcal{N}_\alpha^-|f|$ lies in $L^p(\mathcal{M})$. This time f is the nontangential limit on $\mathcal{M}$ from Ω^-, denoted by $f^- \in L^p(\mathcal{M})$, and we have that:

$$-f(z) = \frac{1}{4\pi} \int_{\mathcal{M}} \frac{\overline{y-z}}{|y-z|^3} n(y) f^-(y) d\sigma(y), \qquad z \in \Omega^-, \tag{68}$$ Cauchy-

that is to say $f = -\mathfrak{C} f^-$ on Ω^-.

Whenever $h \in L^p(\mathcal{M})$ is $\mathfrak{C}$-valued, then $\mathcal{C}h$ is left Clifford-analytic on $\mathbb{R}^n \setminus \mathcal{M}$, moreover $\mathcal{N}^{\pm}|\mathcal{C}h|$ lies in $L^p(\mathcal{M})$ and $\mathcal{C}h$ has nontangential limits $\mathcal{C}^{\pm}h$ a.e. on $\mathcal{M}$, from Ω^+ and Ω^-. Moreover we have that

$$\mathcal{C}^{\pm}h(y) = \pm\frac{h(y)}{2} + \mathcal{S}\mathcal{C}h(y), \qquad y \in \mathcal{M}, \tag{69}$$ Plemelj

where $\mathcal{S}\mathcal{C}h$ is the *singular Cauchy integral operator* defined by

$$\mathcal{S}\mathcal{C}h(y) = \frac{1}{4\pi}\lim_{\varepsilon\to 0}\int_{\mathcal{M}\setminus B(y,\varepsilon)} \frac{\overline{\xi - y}}{|\xi - y|^3} n(\xi)h(\xi)d\sigma(\xi), \qquad y \in \mathcal{M}; \tag{70}$$

see, *e.g.* [33, Eqn. 4.7.12].
Note the analog of the Plemelj formula:

$$\mathcal{C}^+h(y) - \mathcal{C}^-h(y) = h(y). \tag{71}$$ Plemeljs

secrfc *7.4. Regular families of cones on Lipschitz domains*

For $\mathcal{M}$ a compact hypersurface embedded in $\mathbb{R}^{n+1}$ with the local Lipschitz graph property, recall the notation set up in Section 1.6 regarding G-atlases, coordinate cylinders, natural cones and so on, as well as the definition of a regular family of cones given in Section 4. Throughout this section, the symbol M will indicate the supremum of the Lipschitz constants of the parametrizations of the G-atlas under consideration: $M := \sup_j \mathfrak{c}_{\Psi_j}$. We also use the following notation: if $B \subset \mathbb{R}^n$ is a ball, L an isometry of $\mathbb{R}^{n+1}$ and $\mathcal{Y} = L(B \times [a,b]) \subset \mathbb{R}^{n+1}$ a doubly truncated cylinder with cross section B, we let $\mu\mathcal{Y}$ indicate for any $\mu > 0$ the dilated (or contracted) cylinder $L(\mu B \times [a,b]) \subset \mathbb{R}^{n+1}$ with cross section μB. Hereafter we prove that a regular family of cones does exist. First, we need a lemma.

axpos **Lemma 7.7.** *There exists a continuous map $\nu : \mathcal{M} \to \mathbb{S}^n$ and a G-atlas whose parametrization domains are balls in $\mathbb{R}^n$, as well as $c \in (0,1)$ such that: (i) if $\xi \in \mathcal{M}$ belongs to the domain of a chart with u to denote the direction of the corresponding coordinate cylinder, then $\langle \nu(\xi), u\rangle > c$; (ii) if ξ is a regular point, then $\langle \nu(\xi), n_\xi\rangle > c$.*

Proof. Let $\cup_{j=1}^N (U_j, \phi_j)$ be a G-atlas for $\mathcal{M}$. We require without loss of generality that $\mu\mathcal{C}_j$ is still a coordinate cylinder for some $\mu > 1$ and all j. Assume that $U_{j_1} \cap U_{j_2} \neq \emptyset$ with $v^{\pm}_{j_1} \neq v^{\pm}_{j_2}$, and let $\xi \in U_{j_1} \cap U_{j_2}$. Note that $v^-_{j_1} \neq v^+_{j_2}$ otherwise, for $\varepsilon > 0$ small, $\xi + \varepsilon v^-_{j_1} = \xi + \varepsilon v^+_{j_2} \in \Omega^- \cap \Omega^+ = \emptyset$, a contradiction. Let Π be the plane generated by the independent vectors $v_{j_1}^-$ and $v_{j_2}^-$. The intersection $\Pi \cap \mathcal{C}_{1_1} \cap \mathcal{C}_{j_2}$ is a union of parallelepipeds with opposite sides parallel to $v_{j_1}^-$ or $v_{j_2}^-$, and ξ belongs to one of them, say P. Then, $P \cap \mathcal{M}$ is an open arc Γ, with Lipschitz parametrization $L_{j_1}^{-1}(y, \Psi_{j_1}(y))$ as y ranges over an open interval, contained in B_j, of the line $\mathrm{L}_j(\Pi) \cap (\mathbb{R}^n \times \{0\})$. Hence, $\Gamma \setminus \{\xi\}$ has two connected components, each of which must lie in one of the four open cones cut out in P by the two oriented straight lines Δ_1 and Δ_2 through ξ with direction $v_{j_1}^-$ and $v_{j_2}^-$, because $\mathcal{M}$ intersects Δ_1 (resp. Δ_2) only at ξ within P, by definition of coordinate cylinders. Assume that one of the connected components of $\Gamma \setminus \{\xi\}$, say Γ_1, intersects (and therefore is contained in) the open cone $\xi + \mathfrak{C}$, where $\mathfrak{C}$ consists of linear combinations with strictly positive coefficients of $v_{j_1}^-$ and $v_{j_2}^-$. For $\eta > 0$, define $x_{1,\eta} := \xi + \eta v_{j_1}^-$ and $x_{2,\eta} := \xi + \eta v_{j_2}^-$, and consider in Π the closed triangle T with vertices $\{\xi\,, x_{1,\eta}, x_{2,\eta}\}$. For η small enough, T is contained in P and does not contain Γ_1. Because Γ_1 has points interior to T (close to ξ), it must meet the boundary of T on the open segment $(x_{1,\eta}, x_{2,\eta})$ (for it cannot intersect the other two sides). Thus, $T \setminus (\Gamma_1 \cup \{\xi\})$ is disconnected and the sides $[\xi, x_{1,\eta}]$, $[\xi, x_{2,\eta}]$ belong to different components. Now, $x_{1,\eta}$ and $x_{2,\eta}$ both lie in Ω^- for small η, so they can be joined in $P \cap \Omega^-$ by a parametrized Lipschitz curve γ which meets Δ_1 (resp. Δ_2) only at the initial point $x_{1,\varepsilon}$ (resp. endpoint $x_{2,\varepsilon}$), as is obvious from the definiton of the coordinates cylinders $\mathcal{C}_{j_1}$ and $\mathcal{C}_{j_2}$. The curve γ must lie in T, for if it was exterior to T (except for the initial and end points), it would meet Δ_1 and Δ_2 at other points than $x_{1,\varepsilon}$ and $x_{2,\varepsilon}$ because lines separate the plane. But since $x_{1,\varepsilon}$ and $x_{2,\varepsilon}$ lie in different component of $T \setminus (\Gamma_1 \cup \{\xi\})$, of necessity γ meets Γ_1, contradicting the fact that it lies in Ω^-. A similar argument shows that no connected components of $\Gamma \setminus \{\xi\}$ can meet the opposite cone $\xi - \mathfrak{C}$. To sum up, if we set $v_t := t v_{j_1}^- + (1-t) v_{j_2}^-$ with $t \in (0,1)$, then any line parallel to v_t through a point of $\mathcal{M} \cap P$ meets $\mathcal{M}$ only at this point. Next, let $\{Q_0, Q_0 + \lambda_1 v_{j_1}^-, Q_0 + \lambda_1 v_{j_1}^- + \lambda_2 v_{j_2,}^-, Q_0 + \lambda_2 v_{j_2}^-\}$, with $\lambda_1, \lambda_2 > 0$, be the four vertices of P. As $\mu\mathcal{C}_{j_1}$ and $\mu\mathcal{C}_{j_2}$ are coordinate cylinders, the segment $[Q_0, Q_0 + \lambda_1 v_{j_1}^-]$ meets $\mathcal{M}$ at some unique point, say ξ_1. The same argument as before now implies that $\mathcal{M}$ cannot intersect $(\xi_1 + \mathfrak{C}) \cap P$, therefore if $\xi_1 \neq Q_0 + \lambda_1 v_{j_1}^-$ then lines parallel to $v_{j_2}^-$ through a point of $[\xi_1, Q_0 + \lambda_1 v_{j_1}^-]$ could not meet $\mathcal{M} \cap P$, a contradiction since $\mathcal{C}_{j_2}$ is a coordinate cylinder. Swaping the roles of j_1 and j_2, we likewise obtain that $\mathcal{M}$ meets $[Q_0 + \lambda_1 v_{j_1}^- + \lambda_2 v_{j_2,}^-, Q_0 + \lambda_2 v_{j_2}^-]$ in $Q_0 + \lambda_2 v_{j_2}^-$. Thus, $\Gamma (= \mathcal{M} \cap P)$ connects

the two vertices $Q_0 + \lambda_1 v^-_{j_1}$ and $Q_0 + \lambda_2 v^-_{j_2}$, implying that any line parallel to v_t which intersects P must meet Γ. Altogether, we have shown that when $v^-_{j_1} \neq v^-_{j_2}$, then the vector $v_t := t v^-_{j_1} + (1-t) v^-_{j_2}$ is nonzero for every $t \in (0,1)$ and that $v_t/|v_t|$ is the direction of a coordinate cylinder over $U_{j_1} \cap U_{j_2}$; by definition of $v^-_{j_1}$ and $v^-_{j_2}$, this also holds for $t = 0, 1$, and obviously it remains true if $v^-_{j_1} = v^-_{j_2}$.
Proceeding inductively, we deduce that if $U_{j_1} \cap U_{j_2} \cap \cdots \cap U_{j_k} \neq \emptyset$, then any convex combination of the $v^-_{j_\ell}$ is nonzero and, once normalized, to unit norm, defines the direction of a coordinate cylinder over $U_{j_1} \cap U_{j_2} \cap \cdots \cap U_{j_k}$. *We claim* there is a convex combination w of the $v^-_{j_\ell}$ such that $\langle w, v^-_{j_\ell} \rangle > 0$ for $1 \leq \ell \leq k$. To see this, we proceed by induction on k, the case $k = 1$ being trivial. Let $G := (\langle v^-_{j_i}, v^-_{j_\ell} \rangle) \in \mathbb{R}^{k \times k}$ be the Gram matrix of the $v^-_{j_\ell}$. By Farkas's lemma from linear programing [38, Lem. 1], either there is $y \in \mathbb{R}^k$ such that $\langle (-1, 0, \cdots, 0)^t, y \rangle < 0$ and $(G, I_k)^t y$ has only non-negative components, or else there exists $x \in \mathbb{R}^{2k}$ with $x_i \geq 0$ for $1 \leq i \leq 2k$ such that $(G, I_k)x = (-1, 0, \cdots, 0)^t$. However, the latter is impossible for it would imply that $|\sum_{i=1}^k x_i v^-_{j_i}|^2 = -\sum_{i=1}^k x_i x_{k+i} - x_1 \leq 0$, hence $\sum_{i=1}^k x_i v^-_{j_i} = 0$ but we showed that no convex combination of the $v^-_{j_i}$ can be zero, therefore we necessarily have $x_1 = \cdots = x_k = 0$ and thus, $(x_{k+1}, \cdots, x_{2k})^t = (G, I_k)x$ is equal to $(-1, 0, \cdots, 0)^t$, contradicting the fact that $x_{k+1} \geq 0$. Hence, a vector y as above must exist, which means that $y_\ell \geq 0$ for all ℓ and $y_1 > 0$ while Gy has only non-negative components. Then, $w_k := \sum_{\ell=1}^k y_\ell v^-_{j_\ell} / (\sum_{\ell=1}^k y_\ell)$ gives us a convex combination of the $v^-_{j_\ell}$ such that $\langle w_k, v^-_{j_\ell} \rangle \geq 0$ for $1 \leq \ell \leq k$. Note that if $\langle w, v^-_{j_\ell} \rangle = 0$ for all ℓ, then $w = 0$ which is impossible as we just pointed out. Thus, $\langle w_k, v^-_{j_l} \rangle > 0$ for some $l \in \{1, \cdots, k\}$. By induction, there is a convex combination w_{k-1} of the $\{v^-_{j_\ell},\ \ell \neq l\}$ such that $\langle w_{k-1}, v^-_{j_\ell} \rangle > 0$ for $\ell \neq l$. Putting $u = \lambda w_k + w_{k-1}$ for large positive λ, we get $\langle u, v_{j_\ell} \rangle > 0$ for $1 \leq \ell \leq k$, and then $w := u/(1+\lambda)$ gives us the desired convex combination. *This proves the claim.*
To recap, we have shown that whenever $U_{j_1} \cap U_{j_2} \cap \cdots \cap U_{j_k} \neq \emptyset$, there is a coordinate cylinder over $U_{j_1} \cap U_{j_2} \cap \cdots \cap U_{j_k}$ whose direction w has strictly positive scalar product with every $v^-_{j_\ell}$.
Let now (φ_j) be a Lipschitz partition of unity subordinated to the U_j, and set $\nu_1(\xi) := \sum_{j=1}^N \varphi_j(\xi) v^-_j$. Clearly, ν_1 is continuous, and since $\langle v^-_j, n_\xi \rangle$ is equal to $1/\sqrt{1 + |\nabla \Psi_j(y)|^2}$ when $\xi \in \mathrm{Reg} U_j$ and $y = \Phi_j(\xi)$ it holds that $\langle \nu_1(\xi), n_\xi \rangle \geq 1/\sqrt{1 + n^2 M^2}$ for $\xi \in \mathrm{Reg}\mathcal{M}$. We define $\nu := \nu_1/|\nu_1|$.
Pick $\xi \in \mathcal{M}$ and let $j_1, \cdots, j_k$ be the set of $j \in \{1, \cdots, N\}$ such that $\xi \in U_j$.

Let also $j_{k+1}, \cdots, j_l$ be the set of j such that $\xi \in \overline{U}_j \setminus U_j$. For $r = r(\xi)$ sufficiently small, $B(\xi, r) \cap \mathcal{M}$ intersects no U_j with $j \notin \{j_1, \cdots, j_l\}$, and it is contained in $E := (\cap_{\ell=1}^k U_{j_\ell}) \bigcap (\cap_{\ell=k+1}^l \mu U_{j_\ell})$. Since E is an intersection of coordinate cylinders, we know from the previous part of the proof that there is a coordinate cylinder over E with direction w_ξ such that $\langle w_\xi, v_{j_\ell}^- \rangle > 0$ for $1 \leq \ell \leq l$. By restriction, we get a coordinate cylinder over $B(\xi, r) \cap \mathcal{M}$ with direction w_ξ, and since $\varphi_j(\zeta) = 0$ for $j \notin \{j_1, \cdots, j_l\}$ when $\zeta \in B(\xi, r) \cap \mathcal{M}$, we have that $\langle \nu(\zeta), w_\xi \rangle \geq \inf_{1 \leq \ell \leq l} \langle v_{j_\ell}^-, w_\xi \rangle =: c_\xi > 0$. Finally, we may pick a neighborhoud V_ξ of ξ in $\mathcal{M}$, included in $B(\xi, r) \cap \mathcal{M}$, whose projection in the direction w_ξ is a disk. Covering $\mathcal{M}$ with finitely many V_ξ for $1 \leq i \leq N'$, we get the desired atlas, with $c := \min\{1/\sqrt{1 + n^2 M^2}, \min_i c_{\xi_i}\}/\|\nu_1\|_{L^\infty(\mathcal{M})}$. $\square$

For short, we call the collection of open sets (V_j) of $\mathcal{M}$ appearing in the definition of a regular family of cones a *cover* for the family, see Section 4.

exrfc **Lemma 7.8.** *For any $\mu \in (0, 1)$, there exists a regular family of cones associated to a G-atlas whose coordinate cylinders have cross-section a ball and their contraction by μ, when intersected with $\mathcal{M}$, is a cover for the family, while the length of the coordinate cylinders is independent of μ.*

Proof. Choose a G-atlas $\cup_j (U_j, \phi_j)_{1 \leq j \leq N}$, a continuous map $\nu : \mathcal{M} \to \mathbb{S}^n$ and a constant c as in Lemma 7.7. For $\xi \in U_j$ and $r = r(\xi)$ small enough that $B(\phi_j(\xi), r) \subset B_j$, we let $O_j(\xi, r)$ indicate the right circular coordinate cylinder over $\phi_j^{-1}(B(\phi_j(\xi), r) \cap \mathcal{M}$ obtained by restriction of $\mathcal{C}_j$. Since $\mathcal{M}$ is compact, we can find $\delta > 0$ such that, whenever $\xi \in \mathcal{M}$, there exists $j = j(\xi)$ for which $O_j(\xi, \delta) \subset \mathcal{C}_j$. We note for later use that for $r < \delta$:

$$B(\xi, r) \cap \mathcal{M} \subset \phi_j^{-1}(B(\phi_j(\xi), r)) \subset B(\xi, r\sqrt{1 + M^2}) \cap \mathcal{M}, \qquad (72) \quad \text{transd1}$$

where the first and last balls are in $\mathbb{R}^{n+1}$ and the middle one in $\mathbb{R}^n$. Pick $0 < \gamma < c$. By the uniform continuity of ν, there is $\varepsilon_0 > 0$ such that, for every $\xi_0 \in \operatorname{Reg} \mathcal{M}$, we have $\langle \nu(\xi_0), n_\xi \rangle \geq \gamma$ for $\xi \in \operatorname{Reg} \mathcal{M} \cap B(\xi_0, \varepsilon_0)$. Consequently,

$$|\langle v, \nu(\xi_0) \rangle| \leq (1 - \gamma^2)^{1/2} |v|, \qquad v \in T_\xi \mathcal{M}, \quad \xi \in \operatorname{Reg} \mathcal{M} \cap B(\xi_0, \varepsilon_0). \quad (73) \quad \text{normap}$$

Fix $\xi_0 \in \operatorname{Reg} \mathcal{M}$ and let j be such that $O(\xi_0, \delta) \subset U_j$. We assume without loss of generality that $\varepsilon_0 < \delta$, and $L_j = Id$, whence $v_j^- = e_{n+1} := (0, \cdots, 0, 1)^t$. Then, we can write $\xi_0 = (y_0, \Psi_j(y_0))$ with $y_0 \in B_j$, and (72) implies that each

$\xi \in B(\xi_0, \varepsilon_0) \cap \mathcal{M}$ is of the form $\xi = (y, \Psi_j(y))$ with $|y - y_0| < \varepsilon_0$. Hence, σ-a.e. $\xi \in B(\xi_0, \varepsilon_0) \cap \mathcal{M}$ can be written as

$$\xi = \xi_0 + \int_0^{|y-y_0|} D\phi_j^{-1}(y_0 + tu)(u)dt, \qquad (74) \quad \boxed{\texttt{locM}}$$

with $u = (y - y_0)/|y - y_0| \in \mathbb{S}^n$. Note that (74) indeed holds for σ-a.e. $\xi \in B(\xi_0, \varepsilon_0) \cap \mathcal{M}$, because for $\mathcal{H}^n$-a.e. $u \in \mathbb{S}^n$ the map $t \mapsto \phi_j^{-1}(y_0 + tu)$ is absolutely continuous on $[0, \varepsilon_0]$ with derivative $D\phi_j^{-1}(y_0 + tu)(u)$ at a.e. t, by the (version in polar coordinates of the) absolute continuity on lines of Sobolev functions, see for example [56, Thm. 2.1.4]. If $\theta_0 \in (0, \pi/2)$ is such that $\cos\theta_0 = (1-\gamma^2)^{1/2}$, then (73) means that every tangent vector to $\mathcal{M}$ at a point of $\operatorname{Reg}\mathcal{M} \cap B(\xi_0, \varepsilon_0)$ lies in the complement E of the double (untruncated) cone $C_{\theta_0,\nu(\xi_0)}(0) \cup C_{\theta_0,-\nu(\xi_0)}(0)$. In particular, $D\phi_j^{-1}(y_0 + tu)(u) \in E$ for a.e. t and $\mathcal{H}^n$-a.e. u. Moreover, by (20), $D\phi_j^{-1}(y_0+tu)(u)$ belongs to the vector space Π_u generated in $\mathbb{R}^{n+1}$ by the two orthogonal vectors $(u, 0)^t$ and e_{n+1}. Because Π_u is a plane containing 0, the intersection $\Pi_u \cap E$ is either Π_u or a closed double (untruncated) cone $\mathfrak{C} \cup (-\mathfrak{C})$ with vertex 0 and axis orthogonal to the projection $\nu_u(\xi_0)$ of $\nu(\xi_0)$ onto Π_u. Note that $\nu_u(\xi_0) \neq 0$ because $\langle \nu(\xi_0), e_{n+1}\rangle > c$. Since the integrand of (74) lies in Π_u, so does the integral. Therefore, if $\Pi_u \cap E = \Pi_u$ we have that $|\langle \xi - \xi_0, \nu(\xi_0)\rangle| \le (1-\gamma^2)^{1/2}|\xi - \xi_0|$ for σ-a.e. $\xi \in B(\xi_0, \varepsilon_0) \cap \mathcal{M}$. Assume now that $\Pi_u \cap E = \mathfrak{C} \cup (-\mathfrak{C})$. As $\langle \nu_u(\xi_0), e_{n+1}\rangle > c$, the half-line Δ_u emanating from 0 with direction $(u, 0)^t$ is entirely contained in one of the two half-planes cut out in Π_u by the straight line through 0 with direction $\nu_u(\xi_0)$, which is perpendicular to the axis of $\mathfrak{C} \cup (-\mathfrak{C})$. Therefore $D\phi_j^{-1}(y_0 + tu)(u)$, which is of the form $u + \rho e_{n+1}$ with $\rho \in \mathbb{R}$ by (20), either lies in $\mathfrak{C}$ for all $t \in (0, |y - y_0|)$ or in $-\mathfrak{C}$ for all such t. Since $\mathfrak{C}$ and $-\mathfrak{C}$ are closed and convex, the integral in (74) must belong to one of them, hence $|\langle \xi - \xi_0, \nu(\xi_0)\rangle| \le (1-\gamma^2)^{1/2}|\xi - \xi_0|$ again for σ-a.e. $\xi \in B(\xi_0, \varepsilon_0) \cap \mathcal{M}$. By density, this inequality in fact holds for every $\xi \in B(\xi_0, \varepsilon_0) \cap \mathcal{M}$:

$$B(\xi_0, \varepsilon_0) \cap \mathcal{M} \subset \mathbb{R}^{n+1} \setminus \left(C_{\theta_0,\nu(\xi_0)}(\xi_0) \cup C_{\theta_0,-\nu(\xi_0)}(\xi_0)\right), \qquad (75) \quad \boxed{\texttt{empcyl}}$$

because the right-hand side of this inclusion is closed.

Consider now a doubly truncated right circular cylinder Γ_{ξ_0}, with axis having ξ_0 as middle point and direction $\nu(\xi_0)$, whose length $2l$ and radius r_0 satisfy:

$$r_0 < (1-\gamma^2)^{1/2} l/\gamma \qquad \text{and} \qquad l(1-c^2)^{1/2} + r_0 c < \varepsilon_0. \qquad (76) \quad \boxed{\texttt{estimrl}}$$

The exact value of l and r_0 do not matter, as long as they are strictly positive and satisfy (76), but it is important that they can be chosen independently of ξ_0. The first inequality in (76) entails that the initial and final cross sections of Γ_{ξ_0} are included in $C_{\theta_0,\nu(\xi_0)}(\xi_0)\cup C_{\theta_0,-\nu(\xi_0)}(\xi_0)$. Moreover, as $\langle\nu(\xi_0),n_{\xi_0}\rangle > c$, it holds that $\xi_0\pm\epsilon\nu(\xi_0)\in\Omega^{\mp}$ for $\epsilon>0$ small and therefore, by (75), that $C_{\theta_0,\nu(\xi_0)}(\xi_0)\cap B(\xi_0,\varepsilon_0)$ and $C_{\theta_0,-\nu(\xi_0)}(\xi_0)\cap B(\xi_0,\varepsilon_0)$ belong to distinct connected components of $\mathbb{R}^{n+1}\setminus\mathcal{M}$. Hence, since the axis of Γ_{ξ_0} meets $\mathcal{M}$ at ξ_0 only (by(75)) and otherwise meets both $C_{\theta_0,\nu(\xi_0)}(\xi_0)$ and $C_{\theta_0,-\nu(\xi_0)}(\xi_0)$, the initial and final cross sections of Γ_{ξ_0} must lie in distinct connected components of $\mathbb{R}^{n+1}\setminus\mathcal{M}$ as well. Thus, each segment of length $2l$ parallel to the axis in Γ_{ξ_0} intersects both Ω^+ and Ω^- and therefore meets $\mathcal{M}$. *We claim* that such a segment, say Δ, can intersect $\mathcal{M}$ only once. For if $\xi_1=(y_1,\Psi_j(y_1))$ and $\xi_2=(y_2,\Psi_j(y_2))$ are two distinct points in $\Delta\cap\mathcal{M}$, then $y_1,y_2\in B(y_0,\varepsilon_0)$ since the orthogonal projection of Γ_{ξ_0} onto $\mathbb{R}^n\times\{0\}$ is contained in $B(y_0,\varepsilon_0)$, by the second inequality in (76). Thus, we can find $y_2'\in B(y_0,\varepsilon_0)$ such that $\xi_2'=(y_2',\Psi_j(y_2'))$ is arbitrary close to ξ_2 in $\mathcal{M}$ and $\phi_j^{-1}(y_1+tu)$ is differentiable at a.e. $t\in(0,|y_2'-y_1|)$ for $u=(y_2'-y_1)/|y_2'-y_1|$. Then,

$$\xi_2'=\xi_1+\int_0^{|y_2'-y_1|}D\phi_j^{-1}(y_1+tu)(u)dt, \tag{77}$$ locM2

and since $y_1+tu\in B(y_0,\varepsilon_0)$ for $t\in(0,|y_2'-y_1|)$ we can argue as we did on (74), to conclude that $|\langle\xi_2'-\xi_1,\nu(\xi_0)\rangle|\le(1-\gamma^2)^{1/2}|\xi_2'-\xi_1|$.This prevents ξ_2' from being arbitrary close to Δ, a contradiction *which proves the claim.*
What we just showed is that a rotation $\mathfrak{R}_{\xi_0}$ of angle $\cos^{-1}\langle\nu(\xi_0),e_{n+1}\rangle$ in $\mathbb{R}^{n+1}$, such that $\mathfrak{R}_{\xi_0}(\nu(\xi_0))=e_{n+1}$, makes $\mathfrak{R}_{\xi_0}(\Gamma_{\xi_0}\cap\mathcal{M})$ the graph of some function φ_{ξ_0} over its basis. In addition, if $\xi_1,\xi_2\in\Gamma_{\xi_0}$ and we let w be the orthogonal projection of $\xi_1-\xi_2$ onto the hyperplane orthogonal to $\nu(\xi_0)$, we get upon writing $\xi_1=(y_1,\Psi_j(y_1))$ and $\xi_2=(y_2,\Psi_j(y_2))$ with $y_1,y_2\in B(y_0,\varepsilon_0)$ that

$$\begin{aligned}|\xi_1-\xi_2|^2 &= |w|^2+\langle\left(y_1-y_2,\Psi_j(y_1)-\Psi_j(y_2)\right)^t,\,\nu(\xi_0)\rangle^2\\ &\le |w|^2\left(1+\frac{1+\mathfrak{c}_{\Psi_j}^2}{c^2}\right),\end{aligned}$$

since $|w|>|y_1-y_2|c$ because $\langle\nu(\xi_0),e_{n+1}\rangle>c$. Thus, φ_{ξ_0} is Lipschitz with $\mathfrak{c}_{\varphi_{\xi_0}}\le(1+M^2)^{1/2}/c$, so that Γ_{ξ_0} is a coordinate cylinder with direction $\nu(\xi_0)$ over $\Gamma_{\xi_0}\cap\mathcal{M}$. The natural cones $C_{\theta,\pm\nu(\xi_0)}(\xi)$, where $\xi\in\mu\Gamma_{\xi_0}\cap\mathcal{M}$ and $\tan\theta<\mathfrak{c}_{\varphi_{\xi_0}}^{-1}$, can be truncated to some length $\rho>0$ small enough that they are included in Γ_{ξ_0} and therefore in $\Omega^\mp$. In fact, if we restrict to those θ with $\tan\theta < c/(1+M^2)^{1/2}$ (which is less than $1/\mathfrak{c}_{\varphi_{\xi_0}}$ by what precedes), then ρ can be adjusted independently of ξ_0 for it is enough that $\rho < l$ and $\rho c/(1+M^2)^{1/2} < (1-\mu)r_0$. Let us pick $\theta_1 < \theta < \theta_2 < \tan^{-1} c/(1+M^2)^{1/2}$. Obviously, we have that $C_{\theta_1,\pm\nu(\xi_0)}(\xi_0) \subset C_{\theta,\pm\nu(\xi_0)}(\xi_0) \subset C_{\theta_2,\pm\nu(\xi_0)}(\xi_0)$, and shrinking r_0 further if necessary we may ensure by the uniform continuity of ν that

$$C_{\theta_1,\pm\nu(\xi_0)}(\xi) \subset C_{\theta,\pm\nu(\xi)}(\xi) \subset C_{\theta_2,\pm\nu(\xi_0)}(\xi), \qquad \xi \in \mu\Gamma_{\xi_0}. \tag{78}$$ nfC

Note that the shrinking of r_0 involved to get (78) is independent of ξ_0 and depends solely on θ_1, θ_2, and the modulus of continuity of ν.
As ξ_0 ranges over $\operatorname{Reg}\mathcal{M}$ which is dense, the $\mu\Gamma_{\xi_0}$ cover $\mathcal{M}$ because Γ_{ξ_0} contains $B(\xi_0,\mu r_0)$ whose radius is independent of ξ_0. Therefore, we can find finitely many $\Gamma_{\xi_0^1},\cdots,\Gamma_{\xi_0^L}$ such that the $\mu\Gamma_{\xi_0^\ell}$ cover $\mathcal{M}$ to produce a G-atlas whose coordinate cylinders $\Gamma_{\xi_0^\ell}$ have direction $\nu(\xi_0^\ell)$ and whose graphs have Lipschitz constant at most $(1+M^2)^{1/2}/c$. Then, if we put $z(\xi) = -\nu(\xi)$, (78) indicates that $\{C_{\theta,\pm z(\xi)}(\xi)\}$ is a regular family of cones associated to this atlas, with cover $(\mu\Gamma_{\xi_0^\ell}\cap\mathcal{M})$. Finally, note that the common length l of the coordinate cylinders $\Gamma_{\xi_0^\ell}$ and $\mu\Gamma_{\xi_0^\ell}$ is independent of μ. $\square$

pnb **Remark 1.** *With the notation set up the previous proof, to any $x \in \mathcal{M}$ there exists by density a $\xi_0 \in \operatorname{Reg}\mathcal{M}$ such that $x \in B(\xi_0,\mu r_0/2))$, the ball being in $\mathbb{R}^{n+1}$. Then, $\mu\Gamma_{\xi_0}$ together with the $\mu\Gamma_\xi$ for $\xi \in \operatorname{Reg}\mathcal{M}$ and $|\xi-\xi_0| > 3\mu r_0/2$ is a cover of $\mathcal{M}$, from which we can extract a finite subcover $\mu\Gamma_{\xi_0},\mu\Gamma_{\xi_1},\cdots,\mu\Gamma_{\xi_P}$. This shows that, in Lemma 7.8, a fixed $x \in \mathcal{M}$ can always be assumed to lie interior to exactly one member of the cover of the regular family of cones and to the closure of no other.*

Given a regular family of cones $\{C_{\theta,\pm z(\xi)}(\xi)\}$ and a function $h:\Omega^\pm\to\mathbb{R}^k$, we now compare $N^\pm_{\theta,z}h(\xi) := \sup_{x\in C_{\theta,\pm z(\xi)}(\xi)}|h(x)|$ with the nontangential maximal function *truncated at distance* d, given by

$$\mathcal{NT}^{\Omega^\pm}_{\alpha,d}h := \sup_{\substack{x\in R^{\Omega^\pm}_\alpha(\xi)\\ \mathrm{d}(x,\mathcal{M})<d}} |h(x)|,$$

where $R^{\Omega^\pm}_\alpha(\xi)$ was defined in (44). Recall from (47) the notation f^* for the distribution function of $f:\mathcal{M}\to\mathbb{R}$; *i.e*,

$$f^*(\lambda) := \sigma\left(\{\xi\in\mathcal{M}:\ |f(\xi)|>\lambda\}\right), \qquad \lambda\geq 0. \tag{79}$$ distfundef

Recall also the (centered) Hardy-Littlewood maximal function of $\psi \in L^1(\mathcal{M})$, which is the function $\mathfrak{M}\psi : \mathcal{M} \to \mathbb{R}_+ \cup \{+\infty\}$, defined by

$$\mathfrak{M}\psi(\xi) = \sup_{r>0} \frac{1}{\sigma(B(\xi,r)\cap\mathcal{M})} \int_{B(\xi,r)\cap\mathcal{M}} |\psi|\, d\sigma,$$

where $B(\xi, r)$ indicates a ball in $\mathbb{R}^{n+1}$. As $\mathcal{M}$ is a space of homogeneous, type, the following weak 1-1 estimate is well-known to hold [9, Thm. (3.5)]:

$$\sigma(\{\xi \in \mathcal{M} : \mathfrak{M}\psi(\xi) > \lambda\}) \leq \frac{C}{\lambda} \|\psi\|_{L^1(\mathcal{M})}. \qquad (80)$$ w11

compconent2 **Lemma 7.9.** *Let $\{C_{\theta,z^{\pm}(\xi)}(\xi), \xi \in \mathcal{M}\}$ be a regular family of cones. To each $\alpha > 1$ and $p \in [1,\infty]$, there exist $d, C_1 > 0$ depending on α, p and the family $\{C_{\theta,\pm z(\xi)}(\xi)\}$ such that, for every function $h : \Omega^{\pm} \to \mathbb{R}^k$:*

$$(\mathcal{NT}^{\Omega^{\pm}}_{\alpha,d} h)^* \leq C_1\, (N^{\pm}_{\theta,z} h)^*. \qquad (81)$$ compcone2

Proof. We only consider the case of Ω^+, as the proof for Ω^- is similar. Because any regular family of cones can be refined into one of the type described in Lemma 7.8, at the cost perhaps of truncating the cones to smaller length, we may assume that the G-atlas (U_j, ϕ_j) and the regular family of cones $\{C_{\theta,z(\xi)}(\xi), \xi \in \mathcal{M}\}$ are as in this lemma. We use the notation set up in Section 1.6, in particular coordinate cylinders are denoted by $\mathcal{C}_j$, and after the previous remark their cross-section is a ball. We will adapt a density-point argument, much in the style of [33, Prop. 2.1.2]. However, technicalities arise due to the local character of natural cones.

For $x \in \mathcal{C}_j \cap \Omega^+$, we denote by $\xi(x, j) \in U_j$ the projection of x onto $\mathcal{M}$ in the direction v_j^-; this projection uniquely exists, since $\mathcal{C}_j$ is a coordinate cylinder. When $r = r(\xi)$ is small enough that $B(\phi_j(\xi(x, j)), r) \subset B_j$, we let $O_j(\xi, r)$ be the coordinate cylinder over $\phi_j^{-1}(B(\phi_j(\xi(x, j)), r))$ obtained by restriction from $\mathcal{C}_j$. Since the $\mu\mathcal{C}_j$ cover $\mathcal{M}$ which is compact, there are $\delta_0, \delta_1 > 0$ such that, whenever $x \in \mathbb{R}^{n+1}$ satisfies $\mathrm{d}(x, \mathcal{M}) < \delta_0$, there exists $j = j(x)$ for which $O_j(x, \delta_1) \subset \mu\mathcal{C}_j$. Let $l > 0$ be the common length of the cones $C_{\theta,z(\xi)}(\xi)$ in our regular family, and put $\delta := \min\{\delta_1, l\}$ as well as $\delta_2 := \delta/(2+(1+M^2)^{1/2})$. Pick $x \in \Omega^+$ with $\mathrm{d}(x, \mathcal{M}) < \min\{\delta_0, \delta_2\}$ and j such that $O_j(x, \delta_1) \subset \mu\mathcal{C}_j$, whence $O_j(x, \delta) \subset \mu\mathcal{C}_j$. Assume for simplicity that $\mathcal{C}_j$ is oriented along the x_{n+1}-axis, so that L_j is the identity, and write $x = (s, t)^t$ with $s \in \mu B_j$ and $t \in (a_j, \Psi_j(s))$. For $\xi_0 \in \mathcal{M}$ such that $|x - \xi_0| = \mathrm{d}(x, \mathcal{M})$, w get since $|x-\xi_0| < \delta_2 < \delta$ that $\xi_0 \in O(x,\delta) \subset \mu\mathcal{C}_j$, so we can write $\xi_0 = (y_0, \Psi_j(y_0))$ with $y_0 \in \mu B_j$ and $|y_0 - s| < \delta_2$. Hence,

$$\begin{aligned} &0 < \Psi_j(s) - t \le |x-\xi_0| + |\xi_0 - (s, \Psi_j(s))| \\ &< \delta_2 + (1+M^2)^{1/2}|s-y_0| < \delta_2(1+(1+M^2)^{1/2}). \end{aligned} \tag{82}$$

inegtr3

Thus, if we set $\delta_3 := \delta_2/(1+M^2)^{1/2}$, we get from (82) that if $y \in \mathbb{R}^n$ satisfies $|y-s| < \delta_3$, then $y \in \mu B_j$ since $\delta_3 < \delta_1$ and moreover:

$$\begin{aligned} &|(y, \Psi_j(y)) - x| \le |(y, \Psi_j(y) - (s, \Psi_j(s))| + \Psi_j(s) - t \\ &< (1+M^2)^{1/2}\delta_3 + \delta_2(1+(1+M^2)^{1/2}) = \delta < l. \end{aligned} \tag{83}$$

inegtr4

In another connection, for $\xi \in U_j$, let $\widetilde{C}_{\theta_1, v_j^+}(\xi)$ be the *untruncated* positive cone with vertex ξ, direction v_j^+ and aperture $2\theta_1$. If we write $\xi = (y, \Psi_j(y))$, a little geometry shows that $x \in \widetilde{C}_{\theta_1, v_j^+}(\xi)$ as soon as $|y-s| < \tan\theta_1(\Psi_j(s) - t)/(1+M\tan\theta_1)$; this holds for any $\theta_1 \in (0, \pi/2)$, but we mean it for θ_1 as in (49). From this and (83) together with (49), we get that $x \in C_{\theta, z(\xi)}(\xi)$ as soon as $\xi = (y, \Psi_j(y))$ where y satisfies:

$$|y-s| < \min\left(\delta_3\,,\ \frac{\tan\theta_1(\Psi_j(s)-t)}{1+M\tan\theta_1}\right). \tag{84}$$

premb4

Now, if $x \in \mathcal{R}_\alpha^{\Omega^+}(\eta)$ for some $\eta \in \mathcal{M}$, then $|x-\eta| \le \alpha \mathrm{d}(x, \mathcal{M}) \le \alpha(\Psi_j(s) - t)$, and letting $K = \tan\theta_1/(1+M\tan\theta_1)$ we see from (84) that the set A_x of those $\xi \in \mathcal{M}$ such that $x \in C_{\theta, z(\xi)}(\xi)$ contains all $(y, \Psi_j(y)) \in U_j$ for which $|y-s| < \min(\delta_3, K|x-\eta|/\alpha)$. Thus, either $|x-\eta| \ge \alpha\delta_3/K$ and then $\mathrm{d}(x, \mathcal{M}) \ge \delta_3/K$, or else $|x-\eta| < \alpha\delta_3/K$ in which case A_x contains all $(y, \Psi_j(y))$ for which $|y-s| \le K|x-\eta|/\alpha$. In view of (72), this implies:

$$\left(x \in \mathcal{R}_\alpha^{\Omega^+}(\eta) \text{ and } \mathrm{d}(x, \mathcal{M}) < \frac{\delta_3}{K}\right) \Longrightarrow B\left(\xi(x,j), \frac{K|x-\eta|}{\alpha}\right) \cap \mathcal{M} \subset A_x. \tag{85}$$

incb1

Observe also, since $|x-\eta| \le \alpha(\Psi_j(s) - t)$ as pointed out after (84), that (82) and the definitions of δ_2 and δ imply $|x-\eta| < \delta_1$, whence $\eta \in O(x, \delta_1) \subset \mu\mathcal{C}_j$. Thus, we may write $\eta = (\zeta, \Psi_j(\zeta))$ with $\zeta \in \mu B_j$ and we see that

$$|x-\eta| \ge |s-\zeta| \ge |\xi(x,j) - \eta|/(1+M^2)^{1/2}.$$

Therefore, by the triangle inequality, we get that

$$B\left(\xi(x,j), \frac{K|x-\eta|}{\alpha}\right) \cap \mathcal{M} \subset B\left(\eta, \left((1+M^2)^{1/2} + \frac{K}{\alpha}\right)|x-\eta|\right) \cap \mathcal{M}. \tag{86}$$

incb2

Now, for $\lambda > 0$, let $O_\lambda := \{\xi \in \mathcal{M} : N^+_{\theta,z}h(\xi) > \lambda\}$ which is open in $\mathcal{M}$, by the continuity of z. Set $A := \mathcal{M} \setminus O_\lambda$, and define for $\gamma \in (0,1)$:

$$A^*_\gamma := \{\xi \in \mathcal{M} : \sigma(A \cap B(\xi, r)) \geq \gamma\sigma(\mathcal{M} \cap B(\xi, r)), \forall r > 0\}.$$

Next, set $d := \delta_3/K$ and define $\mathcal{O}_{\alpha,\lambda} := \{\xi \in \mathcal{M} : \mathcal{NT}^+_{\alpha,d}h(\xi) > \lambda\}$ which is also open in $\mathcal{M}$. Fix $\eta \in \mathcal{O}_{\alpha,\lambda}$ and pick $x \in R^{\Omega^+}_\alpha(\eta)$ with $\mathrm{d}(x, \mathcal{M}) < d$ such that $|h(x)| > \lambda$. From (85) and (86), we deduce that

$$B\left(\xi(x,j), \frac{K|x-\eta|}{\alpha}\right) \cap \mathcal{M} \subset O_\lambda \cap B\left(\eta, \left((1+M^2)^{1/2} + \frac{K}{\alpha}\right)|x-\eta|\right)$$

and therefore, putting for simplicity $\rho := (1+M^2)^{1/2} + \frac{K}{\alpha}$, we get from (14):

$$\frac{\sigma\left(O_\lambda \cap B\left(\eta, \rho|x-\eta|\right)\right)}{\sigma\left(B\left(\eta, \rho|x-\eta|\right) \cap \mathcal{M}\right)} \geq \frac{\sigma\left(B\left(\xi(x,j), \frac{K|x-\eta|}{\alpha}\right) \cap \mathcal{M}\right)}{\sigma\left(B\left(\eta, \rho|x-\eta|\right) \cap \mathcal{M}\right)} \geq \frac{c'}{C'}\left(\frac{K/\alpha}{\rho}\right)^n.$$

So, letting $r := \rho|x-\eta|$, we obtain upon taking complements that

$$\frac{\sigma(A \cap B(\eta, r))}{\sigma(B(\eta, r) \cap \mathcal{M})} \leq 1 - \frac{c'}{C'}\left(\frac{K/\alpha}{\rho}\right)^n,$$

and if we choose γ such that $1 - \frac{c'}{C'}\left(\frac{K/\alpha}{\rho}\right)^n < \gamma < 1$, we find that $\eta \notin A^*_\gamma$. Thus, with this choice of γ, we have that $\mathcal{O}_{\alpha,\lambda} \subset \mathcal{M} \setminus A^*_\gamma$. Consequently, in view of (80),

$$\begin{aligned} \left(\mathcal{NT}^+_{\alpha,d}h\right)^*(\lambda) &= \sigma(\mathcal{O}_{\alpha,\lambda}) \leq \sigma(\mathcal{M} \setminus A^*_\gamma) = \sigma(\{\xi \in \mathcal{M} : \mathfrak{M}\chi_A(\xi) < \gamma\}) \\ &= \sigma(\{\xi \in \mathcal{M} : \mathfrak{M}\chi_{O_\lambda}(\xi) > 1-\gamma\}) \leq \tfrac{C}{1-\gamma}\sigma(O_\lambda) = C_1\left(N^+_{\theta,z}h\right)^*(\lambda). \end{aligned}$$

This achieves the proof. $\square$

For any $f : \mathcal{M} \to \mathbb{R}$, one has $\|h\|_\infty = \inf\{\lambda : h^*(\lambda) = 0\}$ and $\|h\|^p_{L^p(\mathcal{M})} = p\int_0^\infty t^{p-1}h^*(t)dt$ for $1 \leq p \leq \infty$, see *e.g.* [5, Prop. 1.8]. Hence, Lemma 7.9 entails that $\|\mathcal{NT}^{\Omega^\pm}_{\alpha,d}h\|_p \leq C\|N^\pm_{\theta,z}h\|_p$ for some constant C. When h is harmonic, a stronger estimate holds, as we now show. Recall that a harmonic function on an unbounded domain in $\mathbb{R}^m$, $m \geq 3$, is harmonic at infinity if it tends to zero there [6, Thm. 4.8].

equivNTMF **Lemma 7.10.** *Let $\{C_{\theta,z^{\pm}(\xi)}(\xi), \xi \in \mathcal{M}\}$ be a regular family of cones. To each $\alpha > 1$ and $p \in [1, \infty]$, there exist $C > 0$ depending on α, p and the family $\{C_{\theta,\pm z(\xi)}(\xi)\}$ such that, for every harmonic function $h : \Omega^{\pm} \to \mathbb{R}^k$ (including at infinity in the case of Ω^{-}):*

$$\|\mathcal{N}_{\alpha}^{\Omega^{\pm}} h\|_{L^p(\mathcal{M})} \leq C \|N_{\theta,z}^{\pm} h\|_{L^p(\mathcal{M})}.$$

Proof. In view of Lemma 7.9 and the remark after its proof, it is enough by Hölder's inequality to show that

$$\sup\{|h(x)| : x \in \Omega^{\pm}, \, \mathrm{d}(x, \mathcal{M}) > d\} \leq C' \|N_{\theta,z}^{\pm} h\|_{L^1(\mathcal{M})} \tag{87}$$ inegharNT

where C' depends on $\mathcal{M}$ and the regular family of cones. We shall need that to each $\varepsilon > 0$, there exists a C^{∞}-smooth compact hypersurface $\mathcal{S}^{\pm} \subset \Omega^{\pm}$ such that the coordinate cylinders $\mathcal{C}_j$ of the atlas (U_j, ϕ_j) on $\mathcal{M}$, associated to the family $\{C_{\theta,\pm z(\xi)}(\xi)\}$, are also coordinate cylinders with the smooth graph property on $\mathcal{S}^{\pm}$, and if we represent any $\zeta \in \mathcal{S}^{\pm} \cap \mathcal{C}_j$ as $(y, f_j^{\pm}(y))$ where $y \in B_j$ and $f_j^{\pm} : B_j \to \mathbb{R}$ is a C^{∞}-smooth function, then $|f_j^{\pm}(y) - \psi_j(y)| < \varepsilon$. The existence of $\mathcal{S}^{\pm}$ follows from [46, Thm. 1]. Since the $\mu\mathcal{C}_j$ cover $\mathcal{M}$, note that they cover $\mathcal{S}^{\pm}$ as well. Let us fix $\varepsilon = d/2$, where d is as in Lemma 7.9; without loss of generality, we assume that d is less than the common length of the cones in our regular family. Thus, if one picks $\zeta \in \mu\mathcal{C}_j \cap \mathcal{S}^{\pm}$, one sees from (49) upon writing $\zeta = (y, f_j^{\pm}(y))$ for some j and some $y \in B_j$ that $\zeta \in C_{\theta,z(\xi)}^{\pm}(\xi)$ with $\xi = (y, \Psi_j(y))$, whence $|h(\zeta)| \leq N_{\theta,z}^{\pm}(\xi)$. So, if we put $\nu^{\pm}$ to designate the volume measure on $\mathcal{S}^{\pm}$ and $M_1 := \sup_j \|\nabla f_j^{\pm}\|_{L^{\infty}(\mu B_j)}$, we get in view of (22) that

$$\begin{aligned}
&\textstyle\int_{\mu\mathcal{C}_j \cap \mathcal{S}^{\pm}} |h| d\nu^{\pm} = \int_{\mu B_j} |h(y, f_j^{\pm}(y))| (1 + |\nabla f_j^{\pm}|^2)^{1/2} dm_n(y) \\
&\leq (1 + M_1^2)^{1/2} \textstyle\int_{\mu B_j} N_{\theta,z}^{\pm} h(y, \Psi_j(y)) (1 + |\nabla \Psi_j|^2)^{1/2} dm_n(y) \\
&= (1 + M_1^2)^{1/2} \textstyle\int_{\mu\mathcal{C}_j \cap \mathcal{M}} |h| d\sigma.
\end{aligned}$$

As the $\mu\mathcal{C}_j$ cover $\mathcal{M}$ and $\mathcal{S}^{\pm}$, the previous inequality implies that

$$\|h\|_{L^1_{\nu^{\pm}}(\mathcal{S}^{\pm})} \leq c \|N_{\theta,z}^{\pm} h\|_{L^1(\mathcal{M})} \tag{88}$$ majvalS

where c depends on $\mathcal{M}$, $\mathcal{S}^{\pm}$, and the regular family of cones. To achieve the proof it remains for us to show that, for some C depending on $\mathcal{S}^{\pm}$, we have:

$$\sup\{|h(x)| : x \in \Omega^{\pm}, \, \mathrm{d}(x, \mathcal{M}) > d\} \leq C \|h\|_{L^1_{\nu^{\pm}}(\mathcal{S}^{\pm})}. \tag{89}$$ majvalBV

Let Ω_0 be the interior of $\mathcal{S}^+$ and $G(x, x_0)$ the Green function on Ω_0 with pole at $x_0 \in \Omega_0$. As is well-known, $(n-2)\omega_n G(x, x_0) = 1/|x-x_0|^{n-1} + H(x)$, where H is harmonic in Ω_0 with boundary values $H(\zeta) = -(n-2)\omega_n/|\zeta - x_0|^{n-1}$ for $\zeta \in \mathcal{S}^+$ and ω_n is the surface area of $\mathbb{S}^n$. Since $\mathcal{S}^+$ is C^∞-smooth, classical regularity theory (see for example [40, Thm. 8.3]) implies that $\partial_\nu G(x, x_0)$ is a C^∞-smooth function of $(x, x_0) \in \mathcal{S}^+ \times \Omega_0$, where ∂_ν denotes the exterior normal derivative on $\mathcal{S}^+$. Hence, if we set $E_0 := \{x_0 \in \Omega_0 :\ \mathrm{d}(x_0, \mathcal{S}^+) \geq d/2\}$, then $\partial_\nu G(x, x_0)$ is uniformly bounded on $\mathcal{S}^+ \times E_0$ by some constant C_0 and it follows from the Green formula that $|h(x_0)| \leq C_0 \|h\|_{L^1_{\nu^+}(\mathcal{S}^+)}$ for $x_0 \in E_0$. Because $\{x \in \Omega^+,\ \mathrm{d}(x, \mathcal{M}) > d\} \subset E_1$ by construction, we obtain (89) with superscript "+". The argument for the superscript "-" is similar, but a minor adjustment is needed because Ω^- is unbounded and we cannot directly use [40]. Assuming without loss of generality that $0 \in \Omega^+$, one way to proceed is to introduce the inversion $\mathcal{I}(x) = x/|x|^2$ mapping Ω^- onto a bounded domain $\Omega_1 \subset \mathbb{R}^{n+1}$, and the Kelvin transform $K[h](x) = |x|^{1-n} h(\mathcal{I}(x))$ which is harmonic on Ω_1 [6, Thm. 4.7]. Let $\Omega_2 \subset \Omega^-$ be the exterior of $\mathcal{S}^-$ and put $\Omega_3 := \mathcal{I}(\Omega_2)$, which is a domain with C^∞-smooth boundary $\partial\Omega_3$, such that $\Omega_3, \partial\Omega_3 \subset \Omega_1$. If we denote by $G_2(., x_0)$ and $G_3(., z_0)$ the Green functions of Ω_2 and Ω_3 with poles at x_0 and z_0 respectively, and if we write $\partial_\nu G_2(x, x_0)$ and $\partial_\nu G_3(\xi, z_0)$ for the normal derivatives at $x \in \mathcal{S}^-$ and $\xi \in \partial\Omega_3$, we get from the change of variable formula, since the derivative $D\mathcal{I}(x)$ is a similarity transformation with ratio $1/|x|^2$ [6, Prop. 4.2], that $\partial_\nu G_2(x, x_0) = |x_0|^{1-n}|x|^{-(n+1)}\partial_\nu G_3(\mathcal{I}(x), \mathcal{I}(x_0))$. Now, if we set $E_1 := \{x \in \Omega_2 :\ \mathrm{d}(x, \mathcal{S}^-) \geq d/2\}$, then $\mathcal{I}(E_1) \subset \Omega_3$ is at strictly positive distance from $\partial\Omega_3$, so $\partial_\nu G_3(\xi, z_0)$ is bounded on $\partial\Omega_3 \times \mathcal{I}(E_1)$ by classical regularity theory. Thus, $\partial_\nu G_2(x, x_0)$ is bounded on $\mathcal{S}^- \times E_1$ and the proof can proceed as before. □

7.5. Continuity to the boundary of integrals against harmonic measure

For $\Omega \subset \mathbb{R}^{n+1}$ a bounded open set which is regular for the Dirichlet problem (*i.e.* whose complement is non-thin at every point of its boundary [4, Thm. 7.5.1]), the harmonic measure ω_z^Ω is the Borel probability measure on the boundary $\partial\Omega$ such that, for each continuous function $\varphi : \partial\Omega \to \mathbb{R}$, the function $u_\varphi^\Omega(z) := \int \varphi d\omega_z^\Omega$ is harmonic in Ω, continuous on $\overline{\Omega}$, and coincides with φ on $\partial\Omega$, compare Section 4. If φ is a bounded Borel function on $\partial\Omega$, then u_φ^Ω is still a well-defined harmonic function on Ω, and $\lim_{z\to\xi} u_\varphi^\Omega(z) = \varphi(\xi)$ at every continuity point ξ of φ [4, Cor. 6.6.6]. When φ is merely integrable with respect to ω_z^Ω (this does not depend on $z \in \Omega$), the above-mentioned continuity property can fail [4, Ex. 6.6.18], but it does hold if Ω

is the interior of a connected hypersurface $\mathcal{M}$ with the local Lipschitz graph property, provided that $\varphi \in L^2(\mathcal{M})$ (which implies that $\varphi \in L^1(\mathcal{M}, \omega_z^\Omega)$, by Lemma 4.2). This continuity property will be proven below.
We shall need the well-known connection between harmonic measure and the Perron process to solve the Dirichlet problem. More precisely, for $f : \partial\Omega \to [-\infty, +\infty]$ and $z \in \Omega$, define

$$\begin{aligned} \overline{H}_f^\Omega(z) = \ & \inf\{u(z) : u \text{ superharmonic and bounded below on } \Omega, \\ & \qquad \liminf_{\Omega \ni y \to \xi} u(y) \geq f(\xi), \ \text{all } \xi \in \partial\Omega\}, \\ \underline{H}_f^\Omega(z) = \ & \sup\{v(z) : v \text{ subharmonic and bounded above on } \Omega, \\ & \qquad \limsup_{\Omega \ni y \to \xi} v(y) \leq f(\xi), \ \text{all } \xi \in \partial\Omega\}; \end{aligned} \tag{90}$$ PWB

above, a function which is identically $+\infty$ (resp. $-\infty$) is considered to be superharmonic (resp. subharmonic). Now, for each $z \in \Omega$, we get when $f \in L^1(\mathcal{M}, \omega_z)$ (see [4, Thm. 6.4.6]):

$$\underline{H}_f^\Omega(z) = \overline{H}_f^\Omega(z) = \int f d\omega_z^\Omega. \tag{91}$$ Hf

Equation (91) entails that the exact definition of f on a subset of harmonic measure zero of $\partial\Omega$ has no influence on $\underline{H}_f^\Omega$ nor $\overline{H}_f^\Omega$.
Hereafter, we let $\mathcal{M}$ be a compact connected hypersurface embedded in $\mathbb{R}^{n+1}$ with the local Lipschitz graph property, and use the notation of Section 1.6 regarding G-atlases, coordinate cylinders and natural cones. As in Section 4, we put ω_z^+ to mean $\omega_z^{\Omega^+}$ with Ω^+ the interior of $\mathcal{M}$, and we write u_φ^+ instead of $u_\varphi^{\Omega^+}$.

conthml2 **Lemma 7.11.** *If $\varphi \in L^2(\mathcal{M})$ and φ is continuous at $\xi_0 \in \mathcal{M}$, then*

$$\lim_{\Omega^+ \ni z \to \xi_0} u_\varphi^+(z) = \varphi(\xi_0). \tag{92}$$ contirel

Proof. Replacing φ with $\varphi - \varphi(\xi_0)$, we may assume that $\varphi(\xi_0) = 0$. Let (U_j, ϕ_j) be a G-atlas of $\mathcal{M}$ and V_j an open cover such that $\overline{V}_j \subset U_j$. The coordinate cylinders are of the form $\mathcal{C}_j := L_j^{-1}(B_j \times (a_j, b_j))$, with direction $v_j^- := L_j^{-1}(0, \cdots, 0, 1)^t$. For $\xi \in V_j$ and appropriate $\theta \in (0, \pi/2)$, we denote by $C_{\theta, v_j^+}(\xi) \subset \Omega^+$ the natural cones of aperture 2θ relative to U_j, V_j, *cf.* (19). We also put $M := \max_j \mathfrak{c}_{\Psi_j}$. Choose j_0 such that $\xi_0 \in V_{j_0}$ and assume for simplicity that $L_{j_0} = Id$, whence $v_{j_0}^+ = (0, \cdots, 0, -1)^t$. We set $M_\theta u_\varphi^+(\xi) := \sup_{x\in C_{\theta,v_{j_0}^+}(\xi)} |u_\varphi^+(x)|$. It follows from [10, Thm. 2] that $\int_{V_j} M_\theta u_\varphi^+ d\sigma \leq C_1\|\varphi\|_{L^2(\mathcal{M})}$ for some $C_1 = C_1(\mathcal{M}, V_{j_0}, \theta)$. Pick $r_0 > 0$ small enough that, whenever $0 < r \leq r_0$, the open doubly truncated right circular cylinder $O_{j_0}(\xi_0, r) := B(P_n(\xi_0), r) \times (a_{j_0}, b_{j_0})$, having radius r and axis parallel to $v_{j_0}^-$ passing through ξ_0, has its closure contained in $V_{j_0} \times [a_{j_0}, b_{j_0}]$ and is such that $O_{j_0}(\xi_0, r) \cap \Omega^+$ is starlike about any point on the axis sufficiently close to the base $B(P_n(\xi_0), r) \times \{a_{j_0}\}$. Such starlikeness certainly holds as soon as $r_0 < (b_{j_0} - a_{j_0})/M$. The boundary Σ_r of $O_{j_0}(\xi_0, r) \cap \Omega^+$ can be decomposed into three parts: (i) the base $\Sigma_{r,1} := \overline{B(P_n(\xi_0), r)} \times \{a_{j_0}\}$ contained in Ω^+; (ii) the base $\Sigma_{r,2} := \overline{O_j(\xi_0, r)} \cap \mathcal{M}$, contained in $\mathcal{M}$; (iii) the cylindrical hypersurface $\Sigma_{r,3} := \{(z, t) : z \in S(P_n(\xi_0), r),\ a_{j_0} < t < \Psi_{j_0}(z)\}$, contained in Ω^+. One can see that $O_{j_0}(\xi_0, r) \cap \Omega^+$ has the local Lipschitz graph property. In the rest of the proof, we put for simplicity $O^+(\xi_0, r) := O_{j_0}(\xi_0, r) \cap \Omega^+$

By [10, Thm.2], we have that $\int_{V_{j_0}} |u_\varphi^+(\xi - \eta e_{n+1})|^2 d\sigma(\xi) < c$ for all $\eta \leq \eta_0$ small enough and some constant c. In another connection, points in $\Omega^+ \cap V_{j_0} \times (a_{j_0}, b_{j_0})$ which are not of the form $\xi - \eta e_{n+1}$ for some $\xi \in V_{j_0}$ and some $\eta \leq \eta_0$ remain at distance greater than $\delta > 0$ from $\mathcal{M}$. Hence, as $u_\varphi^+(x)$ is bounded for $\mathrm{d}(x, \mathcal{M}) \geq \delta$, it follows that $\int_{O^+(\xi_0, r_0)} |u_\varphi^+|^2 dm_{n+1} < \infty$ and therefore, by Fubini's theorem, $\int_{\Sigma_{r,3}} |u_\varphi^+|^2 d\mathcal{H}^n < \infty$ for a.e. $r < r_0$. Fix such a r and pick $\varepsilon > 0$, together with $\rho > 0$ so small that $\overline{B(\xi_0, \rho)} \cap \mathcal{M} \subset \Sigma_{r,2}$ and $|\varphi(\xi)| < \varepsilon$ for $\xi \in B(\xi_0, \rho) \cap \mathcal{M}$.

Define a function ψ_1 on Σ_r by letting $\psi_1 = u_\varphi^+$ on $\Sigma_{r,1} \cup \Sigma_{r,3}$ and $\psi_1 = \varphi$ on $\Sigma_{r,2} \setminus B(\xi_0, \rho)$ while $\psi_1 = 0$ on $\Sigma_{r,2} \cap B(\xi_0, \rho)$. Thus, ψ_1 lies in $L^2(\Sigma_r, \mathcal{H}^n \lfloor \Sigma_r)$. For $z \in O^+(\xi_0, r)$, set $v_1(z) := \int_{\Sigma_r} \psi_1 d\omega_z^{O^+(\xi_0, r)}$, where $\omega_z^{O^+(\xi_0, r)}$ denotes harmonic measure on $O^+(\xi_0, r)$. By construction, v_1 is harmonic on $O^+(\xi_0, r)$. If we pick $z_0 \in O^+(\xi_0, r)$ and consider for $z \in O^+(\xi_0, r)$ the Radon-Nykodim derivative $K(z, .) := d\omega_z^{O^+(\xi_0, r)} / d\omega_{z_0}^{O^+(\xi_0, r)}$ which lies in $L^\infty(\Sigma_r, \omega_{z_0}^{O^+(\xi_0, r)})$ by Harnack's inequalities, we may write

$$v_1(z) = \int_{\Sigma_r} \psi(\xi) K(z, \xi) d\omega_{z_0}^{O_j(\xi_0, r)}(\xi) = \int_{\Sigma_{r,1} \cup \Sigma_{r,3} \cup (\Sigma_{r,2} \setminus B(\xi_0, \rho))} \psi(\xi) K(z, \xi) d\omega_{z_0}^{O_j(\xi_0, r)}(\xi), \tag{93}$$

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since ψ_1 vanishes on $\Sigma_{r,2} \cap B(\xi_0, \rho)$. Now, let us choose z_0 so that $O^+(\xi_0, r)$ is starlike about z_0. Then, it follows from [34, Lem. 5] that

$$\|K(z, .)\|_{L^\infty(\Sigma_r \setminus B(\xi_0, \rho),\, \omega_{z_0}^{O^+(\xi_0, r)})} \to 0 \qquad \text{when } z \to \xi_0.$$

Fom this estimate, we obtain on applying the dominated convergence theorem in (93) that $\lim_{O^+(\xi_0,r)\ni z\to\xi_0} v_1(z)=0$.
Next, define $\psi_2:\Sigma_r\to\mathbb{R}$ to be φ on $\Sigma_{r,2}\cap B(\xi_0,\rho)$ and 0 elsewhere on Σ_r. Letting $v_2(z):=\int_{\Sigma_r}\psi_2 d\omega_z^{O^+(\xi_0,r)}$, we find since $\omega_z^{O^+(\xi_0,r)}$ is a probability measure and $|\psi_2|<\varepsilon$ that

$$-\varepsilon<\liminf_{O^+(\xi_0,r)\ni z\to\xi_0} v_2(z)\le\limsup_{O^+(\xi_0,r)\ni z\to\xi_0} v_2(z)<\varepsilon.$$

So, if we put $\psi:=\psi_1+\psi_2$ so that $\psi=u_\varphi^+$ on $\Sigma_{r,1}\cup\Sigma_{r,3}$ and $\psi=\varphi$ on $\Sigma_{r,2}$, we get with $v:=v_1+v_2=\int_{\Sigma_r}\psi d\omega_z^{O^+(\xi_0,r)}$ that

$$-\varepsilon<\liminf_{O^+(\xi_0,r)\ni z\to\xi_0} v(z)\le\limsup_{O^+(\xi_0,r)\ni z\to\xi_0} v(z)<\varepsilon. \tag{94}$$ limv

Now, if u is superharmonic and bounded below on Ω^+ with $\liminf_{\Omega^+\ni y\to\xi} u(y)\ge\varphi(\xi)$ for all $\xi\in\mathcal{M}$, it holds that $u\ge u_\varphi^+$ on Ω^+ by (90) and (91). Therefore $\liminf_{O^+(\xi_0,r)\ni y\to\xi} u(y)\ge\psi(\xi)$ for all $\xi\in\Sigma_r$ and hence, by (90) and (91) again, $u\ge v$ on $O^+(\xi_0,r)$. Infimizing over such u, we deduce that $u_\varphi^+\ge v$ on $O^+(\xi_0,r)$, and a similar argument dealing with subharmonic functions yields that also $u_\varphi^+\le v$ there. Hence, v is the restriction to $O^+(\xi_0,r)$ of u_φ^+, so that (94) implies (92) because $\varepsilon>0$ was arbitrary. □

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